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ON SINGULAR SOLUTIONS

OF

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER IN TWO VARIABLES, AND THE GEOMETRICAL PROPERTIES OF CERTAIN INVARIANTS AND COVARIANTS OF THEIR COMPLETE PRIMITIVES.

A DISSERTATION

PRESENTED TO THE FACULTY OF BRYN MAWR COLLEGE FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

BY

ISABEL MADDISON, B.Sc. (Lond.).

[Reprinted from the *Quarterly Journal of Mathematics,* Vol. XXVIII., 1896].

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LIFE.

My studies in Pure Mathematics, Applied Mathematics, and Physics, in which subjects I was examined for the degree of Doctor of Philosophy, extended over a period of ten years. For four years I studied at the University College of South Wales and Monmouthshire under Principal Viriamu Jones and Professor H. W. Lloyd Tanner. For three years I was a student (exhibitioner) of Girton College, Cambridge, England, where I was a pupil of Mr. Dodds, Mr. Berry, Mr. Whitehead, and Mr. Young, and attended lectures by Professor Cayley, Mr. Webb, and others. I passed the Cambridge Mathematical Tripos Examination in 1892 and was placed in the first class, equal to the 27th Wrangler; in the same year I passed the examinations of the Final Mathematical Honour School at Oxford University. The following year I entered Bryn Mawr College as a graduate student, and studied under Professor Charlotte Angas Scott, Professor James Harkness, and Professor A. Stanley Mackenzie. In 1893 I obtained the degree of Bachelor of Science, with Honours, at the University of London, and was also awarded the resident Mathematical Fellowship at Bryn Mawr College, where I continued my work for a second year. In 1894 I was awarded the Mary E. Garrett European Fellowship, which I held at the University of Göttingen where I attended the lectures of Professors Klein, Hilbert, Burckhardt, and others for two semesters, and took part in the work of the Mathematical Seminary.

My sincere gratitude is due to all the Professors mentioned above for their help and interest, and especially to Professor Scott for her unfailing encouragement and assistance in the preparation of this dissertation.

ISABEL MADDISON.

ON SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND THE GEOMETRICAL PROPERTIES OF CERTAIN INVARIANTS AND COVARIANTS OF THEIR COMPLETE PRIMITIVES.

By Isabel Maddison.

The following chronological list of the principal books and articles dealing with the subject under discussion is given here for convenience of reference. Throughout the paper the number after an author's name refers to this list.

1. Boole, *Treatise on Differential Equations*, Cambridge and London, 1859.

2. Darboux, "Sur la surface des centres de courbure des surfaces algébriques," *Comptes Rendus*, t. LXX, p. 1328, June, 1870.

3. Catalan, "Rémarques sur une Note de M. Darboux relative à la surface des centres de courbure d'une surface algébrique," *Comptes Rendus*, t. LXXI, pp. 50—53, July, 1870.

4. Darboux, "Réponse aux rémarques de M. Catalan," *Comptes Rendus* t. LXXI, p. 265, July, 1870.

5. Cayley, "On the theory of the singular solutions of differential equations of the first order," *Messenger of Mathematics*, New Series, Vol. II, pp. 6—12, January, 1873.

6. Darboux, "Sur les solutions singulières des équations différentielle, ordinaires du premier ordre," *Bulletin des Sciences Mathématiques*, Ser. Is t. IV, pp. 158—176, March, 1873.

7. Cockle, "On Singular Solutions, *Quarterly Journal of Mathematics*, Vol. XII, pp. 305—318, 1873.

8. Casorati, "Alcune formole fondamentali per lo studio delle equazioni algebrico-differenziali di primo ordine e secondo grado tra due variabili ad integrale generale algebrica," *Annali di Matematica*, Ser. II, t. VII, pp. 197—201, December, 1874. French translation in the *Bulletin des Sciences Mathématiques*, Ser. II, t. III, pp. 42 *et. seq.*, 1879.

9. Casorati, "Sulla teoria delle soluzioni singolari delle equazioni differenziali," *Rendiconti del Reale Istituto Lombardo*, Ser. II, Vol. VIII, fasc. XIX, December, 1875.

10. Casorati, "Nuova teoria delle soluzioni singolari delle equazioni differenziali di primo ordine e secondo grado tra due variabili," *Atti della Reale Accademia dei Lincei*, Ser. II, t. III, March, 1876.

11. Casorati, "Sulle soluzioni singolari delle equazioni alle derivate parziali," *Rendiconti del Reale Istituto Lombardo*, Ser. II, Vol. IX, fasc. XIV, July, 1876.

12. Cayley, "On the theory of singular solutions of differential equations of the first order," *Messenger of Mathematics*, New Series, Vol. VI, pp. 23—27, 1877.

13. Casorati, "Concernente la teoria delle soluzioni singolari delle equazioni algebrico-differenziali di primo ordine e secondo grado," *Atti della Reale Accademia dei Lincei*, Ser. III. Vol. III, May, 1879.

14. Casorati, "Una formola fondamentale concernente i discriminanti delle equazioni differenziali e delle loro primitive complete," *Bulletin des Sciences Mathématiques*, Ser. II, t. III, February, 1881.

15. Glaisher, "Examples illustrative of Cayley's Theory of Singular Solutions," *Messenger of Mathematics*, New Series, Vol. XII, pp. 1—14, May, 1882.

16. Johnson (Summary of Cayley's results), *The Analyst*, Vol. IV, pp. 1—4, 1883.

17. Fuchs, "Ueber die Differentialgleichungen deren Integrale feste Verzweigungspunkte besitzen," *Sitzungsberichte der Berliner Akademie*, Bd. 32, pp. 699 *et. seq.*, 1884

18. Schmidt, "Ueber die Singulären Lösungen von Differentialgleichungen erster Ordnung," Giessen, (*Dissertation*), 1884.

19. Torelli, "Contribuzione alla teoria delle equazioni algebrico-differenziali," *Giornale di Matematiche*, Vol. XXIV, pp. 280—289, 1885.

20. Workman, "Theory of Singular Solutions of integrable Differential Equations of the first order," *Quarterly Journal of Mathematics*, Vol. XXII, pp. 175—198, pp. 308—324; 1886-87.

21. Johnson, "Singular Solutions, etc. of Homogeneous Differential Equations," *Messenger of Mathematics*, Vol. XVI, pp. 186—188, 1886.

22. Hill, "On the c-and p-discriminants of Ordinary Integrable Differential Equations of the First Order," *Proc. of London Mathematical Society*, Vol. XIX, pp. 561—589, June, 1888.

23. Goursat, "Sur les solutions singulières des équations différentielles simultanées," *American Journal of Mathematics*, Vol. XI, pp. 329—372, January, 1889.

24. Fine, "Singular Solutions of Ordinary Differential Equations," *American Journal of Mathematics*, Vol. XII, pp. 295—322, July, 1889.

25. Wallenburg, "Beitrag zum Studium der algebraischen Differentialgleichungen erster Ordnung," Halle, (*Dissertation*); *Zeitschrift für Mathematik und Physik*, Bd. XXXV, 1890.

26. Hill, "On Node and Cusp-Loci," *Proc. of London Mathematical Society*, Vol. XXII, pp. 216—236, March, 1891.

27. Hamburger, "Singuläre Lösungen der algebraischen Differentialgleichungen erster Ordnung," *Crelle*, Bd. 112, pp. 205—247, September, 1893.

I. Introduction.

§ 1. THE Theory of Singular Solutions is now generally understood to relate to certain solutions of a rational, integral, algebraic equation of the n^{th} degree in p, $\left(\frac{dy}{dx}\right)$, whose coefficients are rational, integral, algebraic functions of x, y. The primitive of this equation is known to be a rational, integral, algebraic equation of the n^{th} degree in an arbitrary constant, Ω, having coefficients which are functions, not necessarily algebraic, of x, y. A solution of the p-equation which cannot be deduced from the primitive by giving Ω any special value is called a "singular solution"; a solution which can be so deduced is called a "particular integral." (We shall denote this by the letters P.I.).

The singular solution is known to be the envelope of the family of curves represented by the complete primitive; hence the geometrical statement of the above is: A rational, integral, algebraic equation of the n^{th} degree in p, whose coefficients are rational, integral, algebraic functions of x, y, represents a family of curves, not necessarily algebraic, depending on the arbitrary parameter Ω, and such that through every point of the plane there pass n curves of the family. The equation of an envelope of this family of curves satisfies the p-equation and is called a "singular solution"; any particular curve of the family is a "particular integral."

If the coordinates of any particular point be substituted for x, y in the Ω- and p-equations these equations give respectively, the n values of Ω which determine the n curves through the point, and the n values of p which determine the directions of the tangents to the n curves at the point. If the particular point be on the envelope, two of the curves through the point are *consecutive*, and the directions of the tangents to these two curves are *consecutive*, hence each of the equations has a pair of equal roots, that is, the envelope is the locus of points at which the Ω-equation has a pair of equal (consecutive) roots in Ω, and the p-equation has a pair of equal (consecutive) roots in p.

But regarding the equations as binary quantics in $\Omega/1$ and $p/1$ respectively, the discriminant of the Ω-equation is the locus of points at which two values of Ω are equal, and the discriminant of the p-equation is the locus of points at which two values of p are equal; therefore we get, as a general result which will be modified later, *a common factor*

of the Ω- and p-discriminants is a singular solution of the p-equation and gives the envelope of the family of curves represented by it.

Duality of the Ω- and p-equations.

§ 2. The study of the Theory of Singular Solutions may be approached in two ways; we may start with the general Ω-equation and (by eliminating Ω between this and its differential with respect to x) find the corresponding p-equation; or starting with the p-equation we may find the complete primitive, though there is not a general process by which this can be done.

It has been pointed out by Darboux, Cayley and others, that these two methods lead to results which appear at first sight paradoxical. It will be found that if we begin with the Ω-equation and form the Ω-discriminant, this is in general a singular solution, but that if we begin with the p-equation and form the p-discriminant, a certain easily found analytical condition has to be satisfied in order that any factor of this may be a singular solution; that is, the family of curves given by the general Ω-equation has an envelope, that given by the general p-equation has not.

This difficulty was apparently first noticed by Darboux in a note in the *Comptes Rendus*, 1870 (quoted by Mr. Workman, No. 20, p. 176), where, dealing with the equation

$$Ap^2 + Bp + C = 0,$$

in which A, B, C are rational integral algebraic expressions in x, y, he says: "It is usually supposed that in general the curves represented by this differential equation have an envelope, and that this envelope is given by the equation

$$R = B^2 - 4AC = 0.$$

Precisely the contrary happens. In general the curves have no envelope, and the curve $R = 0$ is the cusp-locus."

He goes on to show that the necessary condition to be satisfied in order that $R = 0$ may be the envelope is

$$2A\frac{\partial R}{\partial x} - B\frac{\partial R}{\partial y} = 0,$$

simultaneously with $R = 0$ "which cannot generally happen."

Catalan criticises this statement but gives no satisfactory explanation, and Professor Cayley deals with the question (No. 12), showing that the primitive of the general p-equation

is usually a transcendental family of curves which has in general no envelope (Forsyth, *Treatise on Differential Equations*, p. 36), but Hamburger points out that this does not remove the difficulty and does not deal with the case in which a family of algebraic curves has no envelope, a case of which Serret's equation, $(x+y+\Omega)^2 = (x-y)^3$, is an example. (Ex. 5, § 5, fig. III., B.).

The true explanation lies in the dual relation between the two equations. The Ω- and p-equations represent in different ways the same family of curves, the first giving the aggregate of curves forming the family, and the second the aggregate of *elements of arc* (cp. Clebsch, *Vorlesungen über Geometrie*, Bd. I., Pt. 7, Lindemann) of which the curves are made up. The case then is, as Hamburger remarks, similar to that of a curve given by its point and line equations. Just as the presence of a dp on the curve is the exceptional case when the point equation is considered, and the ordinary case when the line equation is taken, so the non-existence of an envelope is the exceptional case when the family of curves is given by its Ω-equation, and the ordinary case when the p-equation is dealt with.

This is obvious from geometrical considerations. The curves determined by consecutive values of Ω in the general Ω-equation are consecutive curves of the family, and in general intersect in real points; the locus of these points is the envelope, therefore an envelope in general exists. The curve determined by giving p a particular value in the general p-equation is the locus of points of contact of tangents drawn to the family in a fixed direction. The curve given by the consecutive value of p in general intersects this in real points, but the locus of these points is not necessarily the envelope, for in order that it should be the envelope, the direction of the locus would have to be at every point the same as the direction determined by the equal values of p at the point—a condition which is not in general satisfied.

Covariants and Invariants other than the Discriminants.

§ 3. Starting with the general p-equation, Fuchs, Wallenberg, Hamburger and others of the later writers on the subject have, by means of the Theory of Functions, made great advances in the general Theory of Singular Solutions, more especially from the analytical side; their investigations depending in general on expanding functions derived from the p-equation in convergent series. This method has led to

the discovery of many new theorems and has enabled many already known properties to be generalised. The difficulty in applying this method to the simple special cases considered below is that there is no general process by which the integral equation can be derived from the p-equation, and the geometrical properties of the curves represented by it investigated. To avoid this difficulty we adopt the second method and start with the integral equation.

In what follows we deal exclusively with quadratic, cubic and quartic families of algebraic curves. The quadratic family has already been very fully investigated by Professor Casorati and others; but no complete discussion of the cubic and quartic families appears to have been made, though many isolated examples of both these families are to be found in different papers on the subject, and the general equation of the n^{th} degree has received much attention.

Starting with the general integral equation, which has functions of x, y as coefficients of the arbitrary parameter, the p-equation can always be found by eliminating Ω between the Ω-equation and its differential with respect to x. The functions equated to zero in the Ω- and p-equations are binary quantics in $\Omega/1$ and $p/1$ respectively, and the geometrical importance of the discriminants of these quantics has already been pointed out. It is evident that other invariants and covariants of these quantics must represent curves and families of curves related in some special way to the original family. The investigation of certain of these invariants and covariants and their geometrical properties is the principal object of the present paper.

Since the quantics are in different variables, the invariants and covariants formed from the two quantics considered as forming a system have no special significance, hence in the quadratic family the discriminants are the only functions to be considered.

In the cubic family however the Hessian and Cubicovariant of the Ω-quantic at once present themselves as functions representing families—quadratic and cubic respectively—whose geometrical relations to the original family are of considerable interest.

In the quartic family the Hessian of the Ω-quantic is a family, and the quadrinvariant and cubinvariant are loci which are geometrically interesting.

The remainder of the introduction explains the notation used; the way in which the loci of singularities occur in the functions connected with these families; the nature of the

singularities which can appear in *every* curve of the family, that is, which can have a locus; and the combinations of loci which are possible. The quadratic, cubic and quartic families are then discussed under separate headings.

Notation.

§ 4. For our present purpose then it is convenient to use a notation differing very slightly from that used by Casorati, and showing clearly the intimate connection between the Theory of Singular Solutions and the Theory of Invariants.

Let the general equation be

$$U = a\Omega^n + nb\Omega^{n-1} + \frac{n(n-1)}{1.2}c\Omega^{n-2} + \ldots = 0,$$

where n has the values 2, 3, 4, ..., and where a, b, c, ... are rational integral algebraic functions of x, y, linear, quadratic, cubic, etc., as the case may be.

Let the general derived p-equation (*i.e.*, the eliminant of $U=0$ and $\frac{dU}{dx}=0$) be

$$V' = Ap^n + nBp^{n-1} + \frac{n(n-1)}{1.2}Cp^{n-2} + \ldots = 0.$$

It may happen that A, B, ..., have a common factor θ, and when this is removed we get the p-equation in its "proper" form

$$V = \alpha p^n + n\beta p^{n-1} + \frac{n(n-1)}{1.2}\gamma p^{n-2} + \ldots = 0,$$

where $A = \theta\alpha$, $B = \theta\beta$, $C = \theta\gamma$, ...

We shall denote the discriminant and Hessian of U by Δ, H; and the corresponding functions of V by Δ_p, H_p. The discriminant of V' may be denoted by Δ'_p. Since Δ is of degree $2(n-1)$ in the coefficients A, B ... (being the eliminant of two equations of degree $n-1$), we have at once $\Delta'_p = \theta^{2n-2}\Delta_p$.

We shall frequently use as an abreviation "the curve a, Δ, etc." for "the curve represented by $a=0$, $\Delta=0$, etc."; we shall also speak of an equation as being a factor of a quantic, meaning that the part equated to zero is a factor of the quantic. Throughout the examples numerical factors are disregarded.

The Fundamental Relation.

The relation between Δ and Δ_p for the general p-equation was first given by Casorati in a paper entitled *Una Formola Fondamentale concernente i Discriminanti delle Equazioni Differenziali e delle loro Primitive Complete.* (No. 14). It is of the simple form

$$\Delta k^2 = \Delta_p \theta^{2n-2},$$

and is of the greatest importance in the theory.

For the quadratic family, $(n=2)$,

$$k = \begin{vmatrix} a, & b, & c \\ a_x, & b_x, & c_x \\ a_y, & b_y, & c_y \end{vmatrix},$$

but for all other cases k is a complicated expression, different forms for which have been found by Casorati, Brioschi, Torelli and Mr. Workman, who calls k^2 the "tac-discriminant."

Singularities.

§ 5. It has already been shown that the common factor of Δ and Δ_p in the *general* case (in which, since the curves are given by a general point equation, there are no singularities) represents the envelope of the family of curves; it is also evident that in the general case $\theta = 1$, and therefore $\Delta_p = \Delta k^2$. Hence the remaining factor of Δ_p is k^2. Now $\Delta_p = 0$ is the equation of the locus of points at which two values of p are equal, and $\Delta = 0$ is, in this general case, the locus of points at which the two equal values of p belong to two consecutive curves of the family; hence $k = 0$ is the locus of points at which p's belonging to non-consecutive curves of the family are equal, *i.e.*, is the tac-locus of the family of curves. Expressing this in the usual notation, we have, for the general Ω-equation,

$$\Delta = E,\ k = T,\ \Delta'_p = \Delta_p = ET^2,\ \theta = 1.$$

It may moreover happen that at every point on some locus $T_1 = 0$, say, two non-consecutive curves of the family have *r-point contact* in pairs so that $T_1 = 0$ is a locus of r-point contacts. Then T_1 counts once as a tac-locus for each successive pair of contacts, and the occurrences of T_1 as a factor in k, Δ will be found to be T_1^{r-1}, $T_1^{2(r-1)}$, *i.e.*, T may contain a factor T_1^{r-1}, and hence we have the more general expression for the tac-locus $T = T_1^{r-1} T_2^{s-1} \ldots$.

Ex. 1. $\quad U = (x^2 - y^3)\,\Omega^2 + 2\,(y^3 - 1)\,\Omega - x^2 - y^3 = 0.$ Fig. 1.

Here
$$\begin{aligned} \Delta &= x^4 - 2y^3 + 1 = E, \\ k &= xy^2 = TT_1^2, \\ \Delta_p &= x^2y^4\,(x^4 - 2y^3 + 1) = T^2T_1^4E, \\ \theta &= 1, \end{aligned}$$

$y = 0$ is a tac-locus on which the curves have 3-point contact.

Node-Locus.

Now consider what happens when every curve of the family has a node. Through any point on the node-locus a special curve, $\Omega=\omega$, of the family passes twice, therefore if we substitute the coordinates of the point for x, y in the equation of the family, we get an expression which has as a factor $(\Omega-\omega)^2$. Again, at this point the expression $\frac{dU}{\partial x}+p\frac{dU}{\partial y}$ will have $\Omega-\omega$ as a factor. Hence the Ω-eliminant of $U=0$ and $\frac{\partial U}{\partial x}+p\frac{\partial U}{\partial y}=0$ will have a squared factor independent of p at every point of the node-locus. But the Ω-eliminant of $U=0$ and

$$\frac{\partial U}{\partial x}+p\frac{\partial U}{\partial y}=0$$

is the p-equation, hence it follows that the equation of the node-locus is a squared factor independent of p in the p-equation, that is, $\theta=N^2$.

To see how the node-locus makes its appearance in the functions Δ, k, Δ_p, let us consider a family of cubic curves, each curve of which is bipartite and nearly approaching its unipartite form. In this penultimate form the two parts approach each other and touch two nearly coincident branches E_1, E_2 of the envelope. When the nodal form is reached, these branches, E_1 and E_2, actually coincide and form the node-locus, as is shown in the diagrams fig. 2, A and B.

Expressing this symbolically, in the penultimate form Δ is EE_1E_2, when the node is formed Δ becomes EN^2 (where E is the remainder of the envelope).

Again, to investigate the change in Δ_p, we have seen that when a node-locus occurs, the factor $\theta, =N^2$, divides out in the p-equation, and the original Δ'_p becomes $\Delta_p\theta^{2(n-1)}=\Delta_pN^{4(n-1)}$ (cp. § 4), that is, the function which in the penultimate form was

$$\Delta'_p=EE_1E_2T^2,$$

becomes when the node-locus is formed $\Delta_pN^{4(n-1)}$.

But as we have shown above Δ_p must be the locus of (i) equal p's belonging to consecutive curves, (ii) equal p's belonging to non-consecutive curves; therefore $\Delta_p=ET'^2$, where T' is the new tac-locus, hence

$$Lt.\,EE_1E_2T^2=EN^2T^2=ET'^2N^{4(n-1)},$$

i.e. $$T^2=T'^2N^{4n-6},$$

i.e. $$T=T'N^{2n-3}.$$

which shows that $2n-3$ branches of the tac-locus coincide with the node-locus and cease to be tac-loci.

Note.—As an instance of this take the quadratic family, $n=2$. In this case $T=T'N$ and one branch of the tac-locus coincides with the node-locus. The nearly coincident branches E_1, E_2 of the envelope bound a region in which no real curves lie (§ 18), in this region there are two imaginary curves through every point and there is in general a real branch T_2 of the tac-locus giving the locus of points at which these imaginary curves touch. When E_1 and E_2 coincide and become N, T_2 coincides with them.

With regard to k it was shown above that $k=T$, and therefore we have at once that in this case $k=T'N^{2n-3}$; or the same thing follows at once from the fundamental relation. The proof for the case of the acnode is exactly similar.

The results then are, writing T for T',

$$\Delta = EN^2,\ k = TN^{2n-3},\ \Delta_p = ET^2,\ \theta = N^2.$$

Ex. 2. $U = \Omega^2 + 2(x^2+y)\Omega + 2x^2y - x^2 + y^2 + a = 0.$ Fig. 2, A.

$$\Delta = x^4 + x^2 - a = EE_1E_2,$$

$$k = (2x^2+1)x = T_1T_2,$$

$$\Delta_p = (x^4+x^2-a)(2x^2+1)^2x^2 = EE_1E_2T_1^2T_2^2,$$

$$\theta = 1.$$

Here $T_2\,(x=0)$ is the tac-locus of an imaginary set of curves lying between E_1, E_2, which, when a is small, are two nearly coincident, real branches of the envelope.

Ex. 3. $U = \Omega^2 + 2(x^2+y)\Omega + 2x^2y - x^2 + y^2 = 0.$ Fig. 2, B.

$$\Delta = (x^2+1)x^2 = EN^2 = 0,$$

$$k = (2x^2+1)x = T_1N,$$

$$\Delta_p = (x^2+1)(2x^2+1)^2 = ET_1^2,$$

$$\theta = x^2 = N^2.$$

This example is formed from the preceding by putting $a=0$. E_1, E_2 and T_2 coincide in $x=0$ and form the node-locus.

Cusp-Locus.

The cusp-locus may be regarded as formed by the evanescence of a series of loops on consecutive curves; see fig. 3, A, B.

In the penultimate form there is a node-locus, N_1, and an envelope, E_1, nearly coincident with it; when the cusp is formed these coincide, forming the cusp-locus C,

i.e. $$\Delta = Lt.\,EE_1N_1^2 = EC^3.$$

We now want to find the occurrence of C in θ. It must be the same for any value of n, so we may take $n=2$; in this

quadratic family there must be one branch of the tac-locus, T_1, say, between E_1 and N_1, which ultimately coincides with N_1 and E_1 to form C, and there cannot be more than one, therefore

$$\Delta'_p = N_1^4 . Lt . ET^2 . E_1 T_1^2 = C^4 (ET^2 C^3) = ET^2 C^7.$$

But $\Delta'_p = \Delta_p \theta^2$, and by Professor Cayley's well-known result for this special case, $\Delta_p = ET^2 C$.

Hence $ET^2 C\theta^2 = ET^2 C^7$, *i.e.*, $\theta = C^3$, and this holds for any value of n.

From θ, Δ, and Δ_p and the fundamental relation, we at once get $k^2 = T^2 C^{6n-8}$.

Collecting all the results, we have

$$\Delta = EN^2 C^3,\ k = TN^{2n-3} C^{3n-4},\ \Delta_p = ECT^2,\ \theta = N^2 C^3.$$

Note.—A simple analytical proof of the occurrence of the cusp-locus cubed in θ for the quadratic family follows at once from § 10.

Ex. 4. $U = \Omega^2 + 2y\Omega + y^2 - x^2(x+a) = 0$. Fig. 3, A.

$$\begin{aligned} \Delta &= (x+a)\,x^2 = E_1 N_1^2, \\ k &= (3x+2a)\,x = T_1 N_1, \\ \Delta_p &= (x+a)(3x+2a)^2 = E_1 T_1^2, \\ \theta &= x^2 = N_1^2. \end{aligned}$$

When a is small $x + a = 0$ is a branch (E_1) of the envelope nearly coincident with N_1 $(x = 0)$ and T_1 $(3x + 2a = 0)$. Putting $a = 0$, we get the following example.

Ex. 5. $U = \Omega^2 + 2y\Omega + y^2 - x^3 = 0$. Fig. 3, B,

$$\begin{aligned} \Delta &= x^3 = C^3, \\ k &= x^2 = C^2, \\ \Delta_p &= x = C, \\ \theta &= x^3 = C^3. \end{aligned}$$

Comparing this with the above, we see that E_1, N_1 and T_1 have coincided in C.

Limit to the possible singularities.

§ 6. In any family of curves the order of the multiple points which occur in every curve of the family and have a locus is limited by the degree of the family. Take for instance the quadratic family

$$a\Omega^2 + 2b\Omega + c = 0,$$

which is also represented by

$$\alpha p^2 + 2\beta p + \gamma = 0.$$

Let a, b, c be m-ics. Then on any particular curve of the family there may happen to be a multiple point of order r, ($r < m$, or $= m$, if the curve is degenerate), but there cannot be a *locus* of such points if $r > 2$. For suppose a particular

curve has a multiple point of order $r > 2$ at any point of the plane. Then at the point there are r branches and r values of p, therefore the p-equation becomes illusory, *i.e.*, the curves $\alpha = 0$, $\beta = 0$, $\gamma = 0$ all pass through the point. Now the curves $\alpha = 0$, $\beta = 0$, $\gamma = 0$ can have only a finite number of points in common, therefore there cannot be a locus of multiple points of order higher than two in the quadratic family. (Here a locus of proper multiple points has been considered; the case of a curve occurring as a repeated branch of any particular curve or curves of the family is different, and must be discussed in connection with particular integrals). Similarly, in the cubic family there cannot be a locus of multiple points of order higher than three.

It will then be seen that the highest singularity that can occur in all the curves of a quadratic family is of the form

$$(y - x^\nu)^2 = x^\mu,$$

and the equation to the family can be written

$$(\xi\Omega + \eta)^2 = (a'\Omega^2 + 2b'\Omega + c')\chi^\mu,$$

ξ, η, χ, a', ... being functions of x, y; χ being the locus of the singularity. The penultimate form of the singularity for μ even is a series of $\frac{1}{2}\mu$ nodes on a curve of finite curvature; and for μ odd is a loop followed by a series of $\frac{1}{2}(\mu - 1)$ nodes on a curve of finite curvature; see U_0, U_{-1} in fig. 5, A.

Again (leaving out of consideration particular integrals) it follows from what has just been said, that in the quadratic family no two loci can coincide. For all the loci discussed—namely, the envelope, tac-locus, and the loci of nodes, cusps, and higher singularities—depend for their existence on the fact that two directions are determined on them by the p-equation, and since the p-equation is of the second degree, it is impossible that more than two directions should be determined at every point of a locus.

On the cubic family, however, certain coincidences of the loci are possible, *e.g.*, we can have a node-locus coinciding with part or all of the envelope, *i.e.*, at the node one branch touches the envelope, the other cuts it (fig. 15, A), and it is easily seen that any combination of the loci which requires the determination of three directions at a point can occur in a cubic family. Similarly we see that in an n-ic family we can have a locus of n-ple points or a combination of loci involving n directions at every point.

Fixed Points.

§ 7. When we substitute the coordinates of any point ξ, η in $a, b, c, \ldots$, we get an equation of the n^{th} degree in Ω, whose n roots Ω_1, Ω_2, ... (which are not necessarily all different) determine n curves of the family through the point. If, however, $a, b, c \ldots$ all vanish for the values ξ, η of the coordinates, *i.e.*, if the curves $a, b, c, \ldots$ all pass through the point, then every curve of the family passes through the point, *i.e.*, *through any point of the plane either n curves or an infinite number of curves pass.*

Since the number of common intersections of $a, b, c, \ldots$ is finite, the number of points through which an infinite number of curves pass is finite. Such points are called "fixed" or "stationary" points of the family; and it is easy to show that, at these points, there are multiple points on the envelope. For the Ω-discriminant is the eliminant of two equations of degree $n-1$, therefore is homogeneous and of degree $2(n-1)$ in $a, b, c, \ldots$. Take the fixed point as origin; then the terms of lowest degree in $a, b, c, \ldots$ are linear in x, y, and the terms of lowest degree in Δ must be of degree $2n-2$ in x, y. That is, *the fixed point is a multiple point of order* $2(n-1)$ *on the envelope.*

The linear expressions in x, y which are the terms of lowest degree in $U = a\Omega^n + nb\Omega^{n-1} + \ldots = 0$, give the tangents to the different curves at the origin; and since these expressions usually involve Ω to the n^{th} degree, the curves have usually contact in sets of n at the fixed point, hence the fixed point lies on the tac-locus and is in general a multiple point on it.

Number of contacts of curve and envelope.

§ 8. It will be found convenient throughout the discussion to consider $a, b, c, \ldots$ as of the same degree m, say, in x, y. It simplifies arguments which depend on the degree of the equations, and it is always legitimate to consider the line infinity as forming part of any of the curves that are of lower degree than m. If we adopt this convention we can find at once the number of points in which each curve of the family touches its envelope.

Let the curves $a, b, c, \ldots$ be of degree m. The Ω-discriminant is the eliminant of the equations

$$X = a\Omega^{n-1} + (n-1)\,b\Omega^{n-2} + \frac{(n-1)(n-2)}{1.2}\,c\Omega^{n-3} + \ldots = 0;$$

$$Y = b\Omega^{n-1} + (n-1)\,c\Omega^{n-2} + \frac{(n-1)(n-2)}{1.2}\,d\Omega^{n-3} + \ldots = 0;$$

therefore, taking the same value of Ω in these two equations, the curves represented by them intersect in m^2 points on the discriminant. The curve $U, = \Omega X + Y, = 0$ passes through these m^2 intersections and has two points in common with the discriminant at each, that is, $U=0$ *in general touches the discriminant in* m^2 *points*, *e.g.*, in a family of conics each conic in general touches its envelope in four points real or imaginary.

If the curves X, Y have any fixed points in common, these are fixed points on the family and multiple points on the discriminant, as was seen above, and all the curves $U=0$ pass through them. This reduces the number of contacts, as does also the presence of singularities and particular integrals.

Particular Integrals.

A discussion of the occurrence of P.I.'s as factors of Δ, Δ_p, θ, k in the general case would be very laborious. It is here attempted only for the quadratic family, and will be found under that heading.

II. The Quadratic Family.

Notation.

§ 9. The principal functions met with in the quadratic family are as follows:—

(1) The general Ω-equation, $U = a\Omega^2 + 2b\Omega + c = 0$, where a, b, c are algebraic functions of x, y.

(2) The Ω-discriminant, $\Delta = ac - b^2$.

(3) The unreduced p-equation, $V' = 0$, where V' is the eliminant of Ω between $U=0$ and $\dfrac{dU}{dx} = 0$, that is,

$$V' \equiv \left(c\frac{da}{dx} - a\frac{dc}{dx}\right)^2 - 4\left(a\frac{db}{dx} - b\frac{da}{dx}\right)\left(b\frac{dc}{dx} - c\frac{db}{dx}\right).$$

Writing $a_x + pa_y$ for $\dfrac{da}{dx}$, &c., we have

$$V' \equiv \{ca_x - ac_x + p(ca_y - ac_y)\}^2$$
$$- 4\{ab_x - ba_x + p(ab_y - ba_y)\}\{bc_x - cb_x + p(bc_y - cb_y)\}.$$

To simplify this expression we make use of the function k already mentioned (§ 4).

(4) The function k (which, it will be seen, is the Jacobian of the curves a, b, c) is the determinant

$$k = \begin{vmatrix} a\ , & b\ , & c \\ a_x, & b_x, & c_x \\ a_y, & b_y, & c_y \end{vmatrix}.$$

Let the minor of a in this determinant be denoted by $\bar{a}$, &c. Then we may write

$$V' \equiv (\bar{b}_y - p\bar{b}_x)^2 - 4(\bar{c}_y - p\bar{c}_x)(\bar{a}_y - p\bar{a}_x).$$

Comparing this with the standard form, *i.e.*, with

$$V' \equiv Ap^2 + 2Bp + C = 0,$$

we get

$$A = \bar{b}_x^2 - 4\bar{a}_x\bar{c}_x,$$
$$B = -\bar{b}_x\bar{b}_y + 2\bar{a}_x\bar{c}_y + 2\bar{c}_x\bar{a}_y,$$
$$C = \bar{b}_y^2 - 4\bar{a}_y\bar{c}_y.$$

Note.—It is by means of these formulæ that the differential equation is most readily found from the integral equation in any special case.

(5) The unreduced p-discriminant, $\Delta_p' = AC - B^2$, is found from the above relations to be

$$\Delta_p = (\bar{b}_x^2 - 4\bar{a}_x\bar{c}_x)(\bar{b}_y^2 - 4\bar{a}_y\bar{c}_y) - (-\bar{b}_x\bar{b}_y + 2\bar{a}_x\bar{c}_y + 2\bar{c}_x\bar{a}_y)^2,$$
$$= 4(\bar{a}_x\bar{b}_y - \bar{a}_y\bar{b}_x)(\bar{b}_x\bar{c}_y - \bar{b}_y\bar{c}_x) - 4(\bar{a}_x\bar{c}_y - \bar{c}_x\bar{a}_y)^2.$$

But by a known property of determinants

$$\bar{b}_x\bar{c}_y - \bar{b}_y\bar{c}_x = ak, \text{ \&c.};$$

therefore $$\Delta'_p = 4k^2(ac - b^2),$$

i.e. $$\Delta'_p = 4k^2\Delta \text{ ....(I).}$$

(6) It may happen that A, B, C have a common factor, θ, so that $A = \theta\alpha$, $B = \theta\beta$, $C = \theta\gamma$. The p-equation found by dividing V' by this factor is called the "reduced p-equation," and is denoted by V, that is,

$$V = \alpha p^2 + 2\beta p + \gamma = 0.$$

(7) The reduced p-discriminant is Δ_p, where

$$\Delta_p = \alpha\gamma - \beta^2,$$
$$= \Delta'_p / \theta^2,$$

i.e. $$\Delta'_p = \theta^2\Delta_p.$$

(8) Substituting in (I), we get Casorati's fundamental formula for the case $n = 2$ (§ 4), viz.

$$\Delta k^2 = \theta^2 \Delta_p,$$

numerical factors being disregarded.

Note.—The notation just explained is founded on that used by Professor Casorati (No. 8), but differs from it as follows :—

x, y	are	written	for	Casorati's	u, v;
Δ	is	„	„	„	g,
Δ'_p	„	„	„	„	G,
Δ_p	„	„	„	„	S.

Casorati's Results.

§ 10. Some of the results derived by Professor Casorati from the fundamental formula may be quoted here for convenience of reference.

The p-equation can readily be expressed in terms of Δ, the relation being

$$\theta(\alpha p^2 + 2\beta p + \gamma) = \left(\frac{d\Delta}{dx}\right)^2 - 4\Delta\left(\frac{da}{dx} \cdot \frac{dc}{dx} - \frac{db^2}{dx}\right) \ldots \text{(II)};$$

and since

$$\begin{aligned} ca - 2bb + ac &= 2\Delta, \\ ca_x - 2bb_x + ac_x &= \Delta_x, \\ ca_y - 2bb_y + ac_y &= \Delta_y, \end{aligned}$$

we have

$$ck = \begin{vmatrix} 2\Delta, & b, & c \\ \Delta_x, & b_x, & c_x \\ \Delta_y, & b_y, & c_y \end{vmatrix}, \quad -2bk = \begin{vmatrix} a, & 2\Delta, & c \\ a_x, & \Delta_x, & c_x \\ a_y, & \Delta_y, & c_y \end{vmatrix}, \quad ak = \begin{vmatrix} a, & b, & 2\Delta \\ a_x, & b_x, & \Delta_x \\ a_y, & b_y, & \Delta_y \end{vmatrix}.$$

The relation (II) shows that if θ has a factor in common with Δ, it also has a factor in common with $\dfrac{d\Delta}{dx}$. Hence, "a non-repeated factor of Δ does not divide θ," and "a repeated factor of Δ is repeated at least as often in θ."

From the last set of formulæ we see that "a μ-fold factor of Δ is at least a $\mu - 1$-fold factor of k."

Professor Casorati then enumerates the possible arrangements of factors of Δ, Δ_p, under the types $q, r, s, \ldots$, where

$$\begin{aligned} \Delta &= q \ldots r \quad \ldots s^{2\mu+1} \ldots t^{2\nu+1} \ldots u^{2\xi} \ldots v^0 \ldots w^{2\zeta}, \\ \Delta_p &= q \ldots r^{2\lambda+1} \ldots s \quad \ldots t^{2\rho+1} \ldots u^0 \ldots v^{2\eta} \ldots w^{2\omega}, \end{aligned}$$

and gives their significations as follows :—

(1) Factors q can only be singular solutions: they are not factors of θ or k.

(2) Factors r always give P.I.'s: they appear in k but not in θ.

(3) Factors s never give solutions: they appear in θ and k.

(4) Factors t do not in general give solutions, when they do give them they are P.I.'s: they appear in θ and in k.

(5) Factors u never give solutions: they appear in θ and in k.

(6) Factors v do not in general give solutions, and those which they do give are P.I.'s: they appear in k but never in θ.

(7) Factors w do not in general give solutions, and when they do give them they are P.I.'s: they appear in θ and in k.

These theorems are quoted by Mr. Workman (No. 20, p. 181). In his *Criticism of Casorati's Results* (p. 182) Mr. Workman seems to have lost sight of the fact that the theorems relate merely to a *quadratic* family of curves. He describes the scheme as "inadequate as a classification of the higher singularities," and complains that "no notice is taken of the fact that one curve may serve several functions." Since in the quadratic family (cp. § 6) the higher singularities of which there are loci can only belong to a very special class, it would be impossible to give a general discussion of higher singularities without considering the cubic, quartic, ..., n-ic families. Moreover, as was shown above (§ 6), it is impossible for two loci to coincide in a quadratic family; and we have to proceed to the cubic family to get the case Mr. Workman mentions—that of the node-locus and envelope coincident.

The Function k.

§ 11. Though Professor Casorati shows what factors appear in θ and k, he does not state or prove the geometrical significance of these functions directly. In the quadratic family we saw (§ 5) that $\theta = N^2C^3$, and $k = NC^2T$. The occurrence of P.I.'s in θ and k has yet to be discussed, and it will be found that the power to which the factor is raised in θ and k is a guide to the interpretation of its geometrical function.

It should be noticed that, in the quadratic family, mk is the Jacobian of the system of curves a, b, c. For if we make

a, b, and c homogeneous, and of the m^{th} degree in x, y, z, we have

$$ma = xa_x + ya_y + za_z,$$
$$mb = xb_x + yb_y + zb_z,$$
$$mc = xc_x + yc_y + zc_z;$$

therefore

$$mk = \begin{vmatrix} xa_x + ya_y + za_z, & xb_x + yb_y + zb_z, & xc_x + yc_y + zc_z \\ a_x, & b_x, & c_x \\ a_y, & b_y, & c \end{vmatrix},$$

$$= z \begin{vmatrix} a_x, & b_x, & c_x \\ a_y, & b_y, & c_y \\ a_z, & b_z, & c_z \end{vmatrix},$$

where we may put $z = 1$. We also see that k is of lower degree than it appears to be; and the value of its degree given by Mr. Workman (*l.c.* p. 185) should be reduced by one.

The properties of the Jacobian are given by Salmon (*Conic Sections*, Art. 388; *Higher Plane Curves*. pp. 150, 160, 357, *et seq.*). It is the locus of points whose polar lines with respect to three curves a, b, c meet in a point; it is the locus of dps of curves of the system $a\Omega^2 + 2b\Omega + c = 0$, and also their tac-locus (Clebsch, *loc. cit.*, Vol. I., p. 382); also, if the three curves are of the same degree and have a common point, this point is a dp on the Jacobian. Again, in the quadratic family of conics, each conic touches its envelope, in general, at four different points. The three pairs of lines joining these points intersect on the Jacobian. Now when two of the points of contact come together, *i.e.*, when the conic has 4-point contact with Δ at any point, this point lies on the Jacobian, and, since k has 12 intersections with Δ, there are 12 such points.

Higher Singularities.

§ 12. It was shown above (§ 6) that the highest singularities whose loci can occur in a quadratic family are those which have, as penultimate forms,

(1) a series of consecutive nodes along a curve,

(2) a loop followed by a series of nodes;

and the equation of the family was found to be

$$(\xi\Omega + \eta)^2 = (a'\Omega^2 + 2b'\Omega + c')\chi^\mu,$$

where ξ, η, χ, a', b', c' are functions of x, y; χ is the locus of the singularity, and μ is even in case (1) and odd in case (2). We proceed to investigate the way in which loci of these kinds present themselves in the functions Δ, k, &c.

The equation may be written

$$(\xi^2 - a'\chi^\mu)\Omega^2 + 2(\xi\eta - b'\chi^\mu)\Omega + \eta^2 - c'\chi^\mu = 0.$$

We find at once

$$\Delta = -\chi^\mu(a'\eta^2 - 2b'\xi\eta + c'\xi^2) + \chi^{2\mu}(a'c' - b'^2).$$

$$k = \begin{vmatrix} \xi^2 - a'\chi^\mu, & \xi\eta - b'\chi^\mu, & \eta^2 - c'\chi^\mu \\ 2\xi\xi_x - a_x'\chi^\mu - \mu a'\chi^{\mu-1}\chi_x, & \xi\eta_x + \eta\xi_x - b_x'\chi^\mu - \mu b'\chi^{\mu-1}\chi_x, & 2\eta\eta_x - c_x'\chi^\mu - \mu c'\chi^{\mu-1}\chi_x \\ 2\xi\xi_y - a_y'\chi^\mu - \mu a'\chi^{\mu-1}\chi_y, & \xi\eta_y + \eta\xi_y - b_y'\chi^\mu - \mu b'\chi^{\mu-1}\chi_y, & 2\eta\eta_y - c_y'\chi^\mu - \mu c'\chi^{\mu-1}\chi_y \end{vmatrix}$$

$$= \chi^{\mu-1}(\ldots);$$

$$\bar{a}_x = -[\eta^2(\xi\eta_y - \eta\xi_y) + \mu\eta\chi^{\mu-1}\chi_y(b'\eta - c'\xi)$$
$$+ \chi^\mu\{\eta(\eta b_y' - b'\eta_y + c'\xi_y - \xi c') + \eta_y(c'\xi - b'\eta)\} + \chi^{2\mu}(b'c_y' - c'b_y')],$$

$\bar{b}_x = $ &c.;

and

$$A = \chi^\mu\{\ldots\},$$
$$B = \chi^\mu\{\ldots\},$$
$$C = \chi^\mu\{\ldots\}.$$

Hence for case (1), λ nodes, $\mu = 2\lambda$:

$$\Delta = \chi^{2\lambda}\{\ldots\},$$
$$k = \chi^{2\lambda-1}\{\ldots\},$$
$$\Delta_p = \chi^{2\lambda-2}\{\ldots\},$$
$$\theta = \chi^{2\lambda}\{\ldots\}.$$

For case (2), a loop followed by λ nodes, $\mu = 2\lambda + 3$:

$$\Delta = \chi^{2\lambda+3}\{\ldots\},$$
$$k = \chi^{2\lambda+2}\{\ldots\},$$
$$\Delta_p = \chi^{2\lambda+1}\{\ldots\},$$
$$\theta = \chi^{2\lambda+3}\{\ldots\}.$$

Ex. 6. $U = \Omega^2 + 2y\Omega + y^2 - x^6 - x^7 = 0.$ Fig. 4.

$\Delta = x^6(x+1)$,

$k = x^5(7x+6)$,

$\Delta_p = x^4(x+1)(7x+6)^2$,

$\theta = x^6$.

Here $x=0$ is an osc-node locus, type (1), $\lambda=3$,

$x+1=0$ is the envelope,

$7x+6=0$ is the tac-locus.

Ex. 7. $U=\Omega^2+2(x^2+y)\Omega+(x^2+y)^2+(x+y^2)(x+y^2+a)^2(x+y^2+b)^2=0.$

Fig. 5, A.

$$\Delta=(x+y^2)(x+y^2+a)^2(x+y^2+b)^2,$$
$$k=(4xy-1)\{5(x+y^2)^2+3(a+b)(x+y^2)+ab\}(x+y^2+a)(x+y^2+b),$$
$$\Delta_p=(x+y^2)(4xy-1)^2\{5(x+y^2)^2+3(a+b)(x+y^2)+ab\}^2,$$
$$\theta=(x+y^2+a)(x+y^2+b).$$

In this example, $x+y^2=0$ is the envelope $x+y^2+a=0$, and $x+y^2+b=0$ are node-loci, and when a, b are small we have the penultimate form of singularity (2) for $\lambda=1$. The remaining factors are tac-loci, viz. $4xy-1=0$ and

$$5(x+y^2)^2+3(a+b)(x+y^2)+ab=0.$$

When we put $a=0$, $b=0$ we get the following example.

Ex. 8. $U=\Omega^2+2(y+x^2)\Omega+(y+x^2)^2+(x+y^2)^5=0.$ Fig. 5, B,

$$\Delta=(x+y^2)^5,$$
$$k=(4xy-1)(x+y^2)^4,$$
$$\Delta_p=(4xy-1)^2(x+y^2)^3,$$
$$\theta=(x+y^2)^5.$$

Here $x+y^2=0$ is the locus of a singularity of type (2), $\lambda=1$. Comparing with the preceding examples we see how its occurrences in Δ, etc. are accounted for by the coalescence of node and tac-loci.

Particular Integrals.

§ 13. From Casorati's theorems, we see that in certain special cases P.I.'s are factors of Δ, k, Δ_p, θ. There are two cases;

(1) the P.I. may be a factor of Δ_p, but not a factor of Δ (Casorati's 6);

(2) the P.I. may be a factor of both Δ and Δ_p (Casorati's 2, 4, and 7).

Let us investigate these more closely.

I. Particular Integrals as branches of distinct curves.

§ 14. When the branch of the P.I. (P, say) is a factor of Δ_p but not a factor of Δ, the two values of p determined by the p-equation at any point of the branch $P=0$ are equal, but the values of Ω determined by the integral equation are different ($=\omega_1$, ω_2, say). Suppose $P=0$ is a branch of the curve U_{ω_1}. Then at the consecutive point on $P=0$ the two directions are again equal, and the Ω's are again $=\omega_1$, ω_2,

therefore part of the curve U_{ω_2} has the same direction at every point as $P=0$, that is $P=0$ is a branch of the curve U_{ω_1} and also a branch of U_{ω_2}; it may moreover happen to be a μ-fold branch of U_{ω_1} and a ν-fold branch of U_{ω_2}.

If now we substitute for Ω from the equation

$$\omega = \frac{\Omega - \omega_1}{\Omega - \omega_2},$$

we get an equation in ω in which $P=0$ is part of the curve for $\omega=\infty$ and $\omega=0$; and the Ω-equation may be written

$$a'P^{\mu}\omega^2 + 2b\omega + c'P^{\nu} = 0,$$

where μ, ν are positive integers, and a', b', c' are supposed not to contain P as a factor.

Then $$\Delta = a'c'P^{\mu+\nu} - b^2,$$

which shews that P cannot be a factor of Δ.

Again

$$k = \begin{vmatrix} a'P^{\mu}, & b, & c'P^{\nu} \\ a'_xP^{\mu} + \mu a'P^{\mu-1}P_x, & b_x, & c'_xP^{\nu} + \nu c'P^{\nu-1}P_x \\ a'_yP^{\mu} + \mu a'P^{\mu-1}P_y, & b_y, & c'_yP^{\nu} + \nu c'P^{\nu-1}P_y \end{vmatrix}$$

$$= P^{\mu+\nu}\begin{vmatrix} a', & b, & c' \\ a'_x, & b_x, & c'_x \\ a'_y, & b_y, & c'_y \end{vmatrix} + \mu a'P^{\mu+\nu-1}\begin{vmatrix} 0, & b, & c' \\ P_x, & b_x, & c'_x \\ P_y, & b_y, & c'_y \end{vmatrix} + \nu c'P^{\mu+\nu-1}\begin{vmatrix} a', & b, & 0 \\ a'_x, & b_x, & P_x \\ a'_y, & b_y, & P_y \end{vmatrix}$$

thus P, in general, occurs in k to the power $\mu+\nu-1$, but in exceptional cases may occur to a higher, say $\mu+\nu+\rho-1$, where ρ may $=0, 1, 2, \ldots$.

From expression (II) for the differential equation (§ 10), we find

$$\theta(\alpha p^2 + 2\beta p + \gamma) = \left[\frac{d}{dx}(a'c'P^{\mu+\nu} - b^2)\right]^2$$
$$-4(a'c'P^{\mu+\nu} - b^2)\left[\frac{d}{dx}\cdot a'P^{\mu}\cdot\frac{d}{dx}c'P^{\nu} - \left(\frac{db}{dx}\right)^2\right]$$
$$= 4\mu\nu a'b^2c'P^{\mu+\nu-2}\left(\frac{dP}{dx}\right)^2 + \text{higher powers of } P.$$

Now μ, ν, a', b, c', $\dfrac{dP}{dx}$ cannot vanish or have P as a factor, therefore the index of P in θ is always $\mu+\nu-2$. By means

of the fundamental formula and the above, we now see that the index of P as a factor of Δ is $2\rho+2$; and denoting by Δ_1, k_1, Δ_{p_1}, θ_1 the indices of P as a factor of the corresponding expressions Δ, k, &c., we have

$$\Delta_1=0,\ k_1=\mu+\nu+\rho-1,\ \Delta_{p_1}=2\rho+2,\ \theta_1=\mu+\nu-2,$$

where ρ may be 0, 1, 2, ...

The geometrical significance of these factors is not very obvious. It may perhaps be explained by supposing that the μ branches $P^{\mu}=0$ of the curve $\omega=\infty$ are consecutive and placed in contact with the ν consecutive branches $P^{\nu}=0$ of $\omega=0$, giving $P=0$ as a tac-locus, while in addition ρ branches of the ordinary tac-locus, or of some other locus (in Example 10, a P. I.), have lost their original function and coincided with P. This would account for the $2\rho+2$ occurrences in Δ_p and for $\rho+1$ of those in k. The remaining $\mu+\nu-2$ factors in k and θ may be due to the $\mu-1$ contacts of the μ branches of the first curve and the $\nu-1$ contacts of the ν branches of the second.

This view is supported by the result in the special case $\nu=0$, *i.e.* in the family

$$U=a'P^{\mu}\Omega^2+2b\Omega+c=0.$$

It will be found that in this case the occurrences of P in the different functions are

$$\Delta_1=0,\ k_1=\mu+\rho-1,\ \Delta_{p_1}=0,\ \theta_1=\mu+\rho-1;$$

where, however, there is a limitation $\rho \not> \mu-1$.

Comparing these with the former results, we see that only those factors that were accounted for by the contact of *different* branches have disappeared. This upholds the theory that the μ branches P are arranged consecutively and give $P^{\mu-1}$ to k and θ as a factor due to their contacts.

Ex. 9.
$$\begin{aligned}
U &= x(x-a)\Omega^2+2y^3\Omega+x(1+\varepsilon y)=0,\\
\Delta &= x^2(x-a)(1+\varepsilon y)-y^4,\\
k &= xy(2x+a\varepsilon y),\\
\Delta_p &= \{x^2(x-a)(1+\varepsilon y)-y^4\}\,x^2y^2(2x+a\varepsilon y)^2.\\
\theta &= 1.
\end{aligned}$$

We have $x=0$ as a P.I. of the type discussed, $\mu=1$, $\nu=1$ and $\rho=0$. If we put $\varepsilon=0$ the tac-locus $2x+a\varepsilon y=0$ loses its original functions and coincides with the P.I. $x=0$, and we have $\mu=1$, $\nu=1$ and $\rho=1$.

Ex. 10.
$$\begin{aligned}
U &= (y+\varepsilon)(y^2+x)\Omega^2+2x\Omega+(y+\varepsilon)(y^2-x)=0,\\
\Delta &= (y+\varepsilon)^2y^4-x^2\{(y+\varepsilon)^2+1\},\\
k &= xy^2(y+\varepsilon),\\
\Delta_p &= x^2y^4(y+\varepsilon)^2\left[(y+\varepsilon)^2y^4-x^2\{(y+\varepsilon)^2+1\}\right],\\
\theta &= 1.
\end{aligned}$$

In this example $y=0$ is a P.I. for the values of Ω given by $\varepsilon\Omega^2+2\Omega-\varepsilon=0$, and $\mu=1$, $\nu=1$, $\rho=1$; $y+\varepsilon=0$ is a P.I. for $\Omega=\infty$ and for $\Omega=0$, and $\mu=1$, $\nu=1$, $\rho=0$.

If we put $\varepsilon=0$ these two P.I.'s coalesce in the one P.I. $y=0$ for $\Omega=\infty$, $\Omega=0$; and we have $\mu=1$, $\nu=1$ and $\rho=2$; showing that a coalescence of two P.I.'s gives rise to an increase in ρ.

II. Particular Integrals as branches of consecutive or coincident curves.

§ 15. When the part of the P.I., which we will now call Q, is a factor both of Δ and Δ_p, then at any point of $Q=0$ the two values of p are equal, and the two values of Ω are equal ($=\omega_1$, say). At the consecutive point on $Q=0$ the two values of p are again equal, and the two values of Ω are ω_1 as before; that is, two curves are given by $\Omega=\omega_1$, each having as a branch, single or repeated, the curve $Q=0$. (For a discussion as to whether the curves are *consecutive* or *coincident*, see note, § 24).

By taking $\omega = \dfrac{1}{\Omega - \omega_1}$, and substituting for Ω in $U=0$, we get an equation in ω. In this equation the two values of ω are infinite at every point in $Q=0$, hence the equation may be written, in its most general form,

$$a' Q^\mu \omega^2 + 2b' Q^\nu \omega + c = 0,$$

where μ, ν are positive integers, and a', b', c are supposed not to contain Q as a factor.

We find that

$$\Delta = Q^\mu a'c - Q^{2\nu} b'^2,$$

and
$$k = \begin{vmatrix} a' Q^\mu, & b' Q^\nu, & c \\ a'_x Q^\mu + \mu a' Q^{\mu-1} Q_x, & b'_x Q^\nu + \nu b' Q^{\nu-1} Q_x, & c_x \\ a'_y Q^\mu + \mu a' Q^{\mu-1} Q_y, & b'_y Q^\nu + \nu b' Q^{\nu-1} Q_y, & c_y \end{vmatrix}$$

$$= Q^{\mu+\nu} \begin{vmatrix} a', & b', & c \\ a'_x, & b'_x, & c_x \\ a'_y, & b'_y, & c_y \end{vmatrix} + \mu a' Q^{\mu+\nu-1} \begin{vmatrix} 0, & b', & 0 \\ Q_x, & b'_x, & c_x \\ Q_y, & b'_y, & c_y \end{vmatrix} + \nu b' Q^{\mu+\nu-1} \begin{vmatrix} a', & 0, & c \\ a'_x, & Q_x, & c_x \\ a'_y, & Q_y, & c_y \end{vmatrix}.$$

Different cases arise according as $\mu \gtreqless 2\nu$.

(1) $\mu > 2\nu$, $\mu = 2\nu + r$, say, where r is a positive integer. In this case $\Delta = Q^{2\nu}(a'cQ^r - b'^2)$; and, since it is assumed that a', b', c have no factor Q, $Q^{2\nu}$ is the highest power of Q in Δ. Again, in general, $Q^{\mu+\nu-1}$ (*i.e.* $Q^{3\nu+r-1}$) is the highest power of Q in k, but it may happen that Q occurs to a higher power, $Q^{3\nu+r+\rho-1}$, say.

To determine the index of Q in θ we use, as above,

expression (II), §10, and writing $\Delta = Q^{2\nu} R$, where $R = Q^r a'c - b'^2$, we have

$$\theta(\alpha p^2 + 2\beta p + \gamma) = \left[(2\nu Q^{2\nu-1} R \frac{dQ}{dx} + Q^{2\nu}\frac{dR}{dx}\right]^2$$

$$- 4Q^{2\nu}R\left[\left\{(2\nu + r) Q^{2\nu+r-1}a'\frac{dQ}{dx} + Q^{2\nu+r}\frac{da'}{dx}\right\}\frac{dc}{dx}\right.$$

$$\left. - \left(\nu Q^{\nu-1}b'\frac{dQ}{dx} + Q^{\nu}\frac{db'}{dx}\right)^2\right],$$

$$= - 4\nu^2 a'b'^2 c Q^{4\nu+r-2}\frac{dQ}{dx} + \text{higher powers of } Q;$$

and since by hypothesis ν, a', b', $c\,\dfrac{dQ}{dx}$ do not vanish or have Q as a factor, we find

$$\theta = Q^{4\nu+r-2}.$$

From the fundamental relation and the above, we can find Δ_p, and the results for $\mu = 2\nu + r$ are:—

$$\Delta_1 = 2\nu,\ k_1 = 3\nu + r + \rho - 1,\ \Delta_{p_1} = 2\rho + 2,\ \theta_1 = 4\nu + r - 2;$$

where $\rho = 0, 1, 2 \ldots$; Δ_1, k_1, &c., denoting the indices of Q as a factor of Δ, k, &c.

(2) $\mu = 2\nu$.

This gives $\qquad \Delta = Q^{2\nu}(a'c - b'^2)$,

Let $a'c - b'^2 = Q^{\lambda}R$, where Q is not a factor of R.

Then $\qquad \Delta = Q^{2\nu+\lambda}R.$

To investigate the factors of k use the expression for ck in terms of Δ (§ 10).

$$ck = \begin{vmatrix} 2Q^{2\nu+\lambda}R, & Q^{\nu}b, & c \\ (2\nu+\lambda)\,Q^{2\nu+\lambda-1}RQ_x + Q^{2\nu+\lambda}R_x, & Q^{\nu}b_x + \nu Q^{\nu-1}bQ_x, & c_x \\ (2\nu+\lambda)\,Q^{2\nu+\lambda-1}RQ_y + Q^{2\nu+\lambda}R_y, & Q^{\nu}b_y + \nu Q^{\nu-1}bQ_y, & c_y \end{vmatrix}$$

$$= Q^{3\nu+\lambda}\begin{vmatrix} 2R, & b, & c \\ R_x, & b_x, & c_x \\ R_y, & b_y, & c_y \end{vmatrix} + \nu b Q^{3\nu+\lambda-1}\begin{vmatrix} 2R, & 0, & c \\ R_x, & Q_x, & c_x \\ R_y, & Q_y, & c_y \end{vmatrix}$$

$$+ (2\nu+\lambda)RQ^{3\nu+\lambda-1}\begin{vmatrix} 0, & b, & c \\ Q_x, & b_x, & c_x \\ Q_y, & b_y, & c_y \end{vmatrix};$$

therefore $Q^{3\nu+\lambda-1}$ is, in general, the highest power of Q which is a factor of k. In exceptional cases $Q^{3\nu+\lambda+\rho-1}$ may be a factor of k.

Again

$$\theta\,(\alpha p^2 + 2\beta p + \gamma) = \left\{(2\nu + \lambda)\ Q^{2\nu+\lambda-1} R\,\frac{dQ}{dx} + Q^{2\nu+\lambda}\,\frac{dR}{dx}\right\}^2$$

$$-\,4\,Q^{2\nu+\lambda} R\left\{\left(2\nu\,Q^{2\nu-1} a'\frac{dQ}{dx} + Q^{2\nu}\frac{da'}{dx}\right)\frac{dc}{dx} - \left(\nu\,Q^{\nu-1} b'\frac{dQ}{dx} + Q^{\nu}\frac{db'}{dx}\right)^2\right\}$$

$$= 4\,Q^{4\nu+\lambda-2} R\nu^2 b'^2 \left(\frac{dQ}{dx}\right)^2 + \text{higher powers of } Q,$$

therefore the highest power of Q in θ is always $Q^{4\nu+\lambda-2}$, and we have for the case $\mu = 2\nu$:

$$\Delta_1 = 2\nu + \lambda,\ k_1 = 3\nu + \lambda + \rho - 1,\ \Delta_{p_1} = \lambda + 2\rho + 2,\ \theta_1 = 4\nu + \lambda - 2\,;$$

where $\rho = 0, 1, 2, \ldots$; $\lambda = 0, 1, 2, \ldots$.

(3) $\mu < 2\nu$, $\mu = 2\nu - s$, where s is a positive integer;

$$\Delta = Q^{2\nu-s}\,(a'c - Q^s b'^2) = Q^{2\nu-s} R \ \text{(say)},$$

where $R = a'c - Q^s b'^2$, and therefore never contains Q as a factor. Hence it follows that Δ_1 always $= 2\nu - s$.

A proof similar to the above shows that, in general, $k_1 = 3\nu - s - 1$, but that k_1 may be greater, namely

$$= 3\nu - s + \rho - 1.$$

Investigating the value of θ_1 as before, we find that $\theta_1 = 4\nu - 2s - 2$ always; the fundamental formula gives Δ_{p_1}, and we have the set of results:

$$\Delta_1 = 2\nu - s,\ k_1 = 3\nu - s + \rho - 1,\ \Delta_{p_1} = s + 2\rho + 2,\ \theta_1 = 4\nu - 2s - 2.$$

The geometrical significance of the repetition of the factor Q in Δ, Δ_p, etc. is not very apparent. For a possible explanation let us go back to the penultimate form and consider

Q^μ as the limit of $(Q + \epsilon_1)\,(Q + \epsilon_2)\ldots\ldots(Q + \epsilon_\mu)$,

and Q^ν " " $(Q + \eta_1)\,(Q + \eta_2)\ldots\ldots(Q + \eta_\nu)$,

where $\epsilon_1, \ldots, \epsilon_\mu, \eta_1, \ldots, \eta_\nu$ are different *small* quantities which vanish in the limit. Then we find that, in general,

the envelope has Δ_1 branches nearly coincident with Q,

and the tac-locus has k_1 " " " Q,

so that, in the limit, when $\epsilon_1, \ldots, \epsilon_\mu, \eta_1, \ldots, \eta_\nu$ vanish,

Δ has as a factor Q^{Δ_1},

k " " Q^{k_1}.

That is :—a certain number (Δ_1) of branches of the envelope have coincided with a P.I. and lost their original function; and a certain number (k_1) of branches of the tac-locus have at the same time coincided with the same P.I. and lost their original function.

Ex. 11.
$$\begin{aligned} U &= y\,(y+\varepsilon)\,\Omega^2 + 2y^2\,(y+\varepsilon-\eta)\,\Omega + x = 0, \\ \Delta &= y\,\{(y+\varepsilon)\,x - y^3\,(y+\varepsilon-\eta)^2\}, \\ k &= y^2\,\{(y+\varepsilon)^2 - \varepsilon\eta\}, \\ \Delta_p &= y^5\,\{(y+\varepsilon)^2 - \varepsilon\eta\}^2\,[(y+\varepsilon)\,x - y^3\,(y+\varepsilon-\eta)^2]. \\ \theta &= 1. \end{aligned}$$

Now $y=0$ is a P.I. for which $\mu=1$, $\nu=2$, $s=3$ and $\rho=0$. When we put $\varepsilon=0$, a branch of the envelope, that is, a branch of

$$(y+\varepsilon)\,x - y^3\,(y+\varepsilon-\eta)^2 = 0,$$

coalesces with the P.I. $y=0$ giving a P.I. for which $\mu=2$, $\nu=2$, $s=2$ and $\rho=1$.

There are exceptions to this in the case (2), $\lambda>1$.

In this case Δ may have a branch $\{Q+f(\varepsilon,\eta)\}^\lambda$, where $f(\varepsilon,\eta)$ vanishes with $\varepsilon_1, \ldots, \varepsilon_\mu, \eta_1, \ldots, \eta_\nu$. This may be a singularity locus, which, when $\varepsilon_1, \ldots, \varepsilon_\mu, \eta_1 \ldots, \eta_\nu$ vanish, coincides with a P.I. and loses its original function. In this case every curve of the family has a stationary singularity at one or more points on $Q=0$.

Ex. 12.
$$\begin{aligned} U &= (x+\varepsilon)^2\,\Omega^2 + 2\,(x+\varepsilon)\,(x+1)\,y\Omega + \{(x+1)^2 + x^3\}\,y^2 = 0. \quad \text{Fig. 6, A}, \\ \Delta &= (x+\varepsilon)^2 x^3 y^2, \\ k &= (x+\varepsilon)^3 x^2 y^2\,(x+3), \\ \Delta_p &= (x+\varepsilon)^4\,xy^2\,(x+3)^2, \\ \theta &= (x+\varepsilon)^2\,x^3 y^2. \end{aligned}$$

In this case $x+\varepsilon=0$ is a P.I. for which $\mu=1$, $\nu=1$, $\lambda=0$, $\rho=1$, and $x=0$ is a cusp-locus.

When we put $\varepsilon=0$, those two loci coalesce and we get $x=0$ as a P.I. with $\mu=2$, $\nu=1$, $\lambda=3$ and $\rho=0$ Fig. 6, B.

The geometrical interpretation of the occurrences in Δ_p and θ is still less obvious. When the values of Δ_1, k_1 have been accounted for, it is possible to have three different pairs of values of Δ_{p1}, θ_1 (corresponding to the cases (1), (2), (3)) related to these; and it is difficult to see what geometrical distinctions are involved by these different sets of values.

Fixed points and Particular Integrals.

§ 16. Considering the two equations just dealt with,

(i) $a'P^\mu\Omega^2 + 2b\Omega + c'P^\nu = 0$,

(ii) $a'Q^\mu\Omega + 2b'Q^\nu\Omega + c = 0$,

we see that in (i) all the curves of the family pass through all the intersections of P and b: in (ii) all the curves of the family pass through all the intersections of Q and c. Hence the P.I.'s, P, Q always pass through some of the stationary points or stationary singularities, but not necessarily through all. In (i) a', b, c' may have common intersections not lying on P, and in (ii) a', b', c may have common intersections not lying on Q, and these will be fixed points not lying on the particular integrals.

Table showing the geometrical properties of any factor whose indices in Δ, k, Δ_p, and θ are known.

§ 17. Before discussing the quadratic families of lines, conics, etc., it will be useful to note the following classification. We have investigated all the different loci which can occur in a quadratic family and found their indices as factors of the four functions Δ, k, Δ_p, and θ. The results can be arranged as follows:—

	Geometrical Signification.	Δ_1	k_1	Δ_{p_1}	θ_1	Casorati's.
1	Ordinary Envelope, Singular Solution.	1	0	1	0	1
2	Tac-locus of r-point contacts.	0	$r-1$	$2(r-1)$	0	6
3	Locus of singularity whose penultimate form is λ adjacent nodes on a curve of finite curvature.	2λ	$2\lambda-1$	$2\lambda-2$	2λ	7
3′	Special case, node-locus.	2	1	0	2	5
4	Locus of singularity whose penultimate form is a loop followed by λ adjacent nodes	$2\lambda+3$	$2\lambda+2$	$2\lambda+1$	$2\lambda+3$	4
4′	Special case, cusp-locus.	3	2	1	3	3
5	P.I.'s belonging to distinct curves $a'P^\mu\Omega^2+2b\Omega+c'P^\nu=0$.	0	$\mu+\nu+\rho-1$	$2\rho+2$	$\mu+\nu-2$	6
6	P.I.'s as parts of coincident or consecutive curves $a'Q^\mu\Omega^2+2b'Q^\nu\Omega+c=0$, $\mu=2\nu+r$,	2ν	$3\nu+r+\rho-1$	$2\rho+2$	$4\nu+r-2$	7
7	$\mu=2\nu$,	$2\nu+\lambda$	$3\nu+\lambda+\rho-1$	$2\rho+2+\lambda$	$4\nu+\lambda-2$	4 or 7
8	$\mu=2\nu-s$.	$2\nu-s$	$3\nu-s+\rho-1$	$2\rho+2+s$	$4\nu-2s-2$	2, 4 or 7
9	P.I. as part of one curve only, $a'P^\mu\Omega^2+2b\Omega+c=0$.	0	$\mu+\rho-1$	0	$\mu+\rho-1$	

Throughout μ, ν, r, s are integers >0,

λ, ρ are integers $\geqq 0$.

No combinations of the different loci can occur and the arrangements are mutually exclusive, except in a special case of 5, namely $\mu=1$, $\nu=1$, which may be confused with 2. *Hence given Δ_1, k_1, Δ_{p_1}, θ_1 in any case, it is possible to determine by means of this table, the geometrical function of the factor considered.*

It will be seen that in 5 (Casorati's 6) the factors usually occur in θ, contrary to Casorati's statement.

Boundary Lines.

§ 18. It is important to notice that in the quadratic family branches of Δ which occur an odd number of times, *i.e.* whose equations appear as factors to an odd power in Δ, form the boundaries of regions in which real curves of the family lie. For as we cross such a branch the sign of Δ changes and therefore (Salmon, *Higher Algebra*, p. 239) the number of pairs of real roots of the Ω-equation changes; that is, there are two real values of Ω at every point on one side of the branch, and two imaginary values of Ω at every point on the other side. This does not hold for the cubic family, for in it a change in the sign of Δ denotes a change from three real roots to one real root, and therefore in the cubic family the curves lie in every part of the plane.

It may, however, happen that a curve given by an imaginary value of Ω has a real branch, and such a branch will lie in the excluded region. It is proved by Crone for the family of conics (*Mathematische Annalen*, Vol. XII., p. 569), that if there is such a branch it must belong to two different curves of the family. It follows therefore from our discussion of P.I.'s that this branch will appear in k and Δ_p.

Ex. 13. $U = x(x+y+1)\Omega^2 + 2xy\Omega + y(x-1) = 0.$ Fig. 7,

$$\Delta = xy(x^2 - y - 1) = 0,$$

$$k = xy(x+1),$$

$$\Delta_p = x^3y^3(x+1)^2(x^2 - y - 1) = 0.$$

$$\theta = 1.$$

The plane is divided into regions by the envelope $x^2 - y - 1 = 0$ and the two P.I.'s $x = 0$, $y = 0$. The line $x + 1 = 0$, which forms part of each of the two imaginary curves $\Omega = -1 \pm i$, lies in the excluded region as shown in the figure.

The Quadratic family of Lines.

§ 19. This family is fully dealt with in such books as Salmon's *Conic Sections*. The functions a, b, c are linear in x and y, and, provided the lines a, b, c do not all pass through one point, their envelope is the conic

$$\Delta = ac - b^2 = 0.$$

If we write $a = a_0x + a_1y + a_2$, $b = \ldots$, $c = \ldots$, then

$$k = \begin{vmatrix} a_0x + a_1y + a_2, & b_0x + b_1y + b_2, & c_0x + c_1y + c_2 \\ a_0 \quad , & b_0, \quad , & c_0 \\ a_1 \quad , & b_1, \quad , & c_1 \end{vmatrix},$$

$$= \begin{vmatrix} a_0, & b_0, & c_0 \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix}.$$

We have already pointed out that the degree of k is lower than that calculated by Mr. Workman (20., p. 185). It is obvious that in this case no tac-locus exists.

The Quadratic family of Conics.

§ 20. This family is discussed by Salmon (*Higher Plane Curves*, Ch. VI.) in his chapter on curves of the fourth order. $\Delta = ac - b^2 = 0$ is a quartic which touches the conics a, c at all their intersections with b. The quartic evidently lies in regions where a and c have the same sign. Every conic of the system has real or quasi-contact with the quartic at the four intersections of the conics $a\Omega + b = 0$, $b\Omega + c = 0$, accounting for the eight intersections of the conic and the quartic.

The equation giving the condition that any conic of the system shall have a dp is of the sixth degree in Ω, therefore six conics of the system reduce to line-pairs. Since each line of the line-pair meets the quartic in four points and these must be coincident in pairs, the line is either a bitangent to the quartic, or touches the curve once and passes through a dp, or joins two dps. Crone (*Mathematische Annalen*, Vol. XII., p. 561) has fully discussed the arrangements of the bitangents for all the different forms of non-singular quartics and has investigated the number of different systems of real conics and the number of isolated dps, etc.

The following theorems with reference to special points in the quadratic family are stated here for the sake of clearness; they are immediate deductions from what has already been proved.

A point through which two, and only two, *different* curves of the family pass does not lie on Δ. This holds even when the point is a dp on one or both of the curves* (cp. § 7).

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* See Ex. 1, § 5. The origin is a point at which $\Omega = 0$, $(x^2 + y^3 = 0)$ and $\Omega = \infty$, $(x^2 - y^3 = 0)$ both have cusps.

A point through which *one* curve of the family passes twice and through which no other curve passes is a dp on Δ. In this case the Ω for the curve is a repeated root of the sextic which gives the six values of Ω for which the conics break up into line-pairs.

Ex. 14. $U = (x-1)\,y\Omega^2 + 2\,(x-1)\,\Omega + x^2 - y^2 = 0,$

$$\Delta = (x-1)\,(x^2y - y^3 - x + 1).$$

The cubic factor is the envelope, and $x - 1 = 0$ is a P.I. The dp $(1, 0)$ is a point through which only the curve $\Omega = \infty$ or $(x-1)\,y = 0$ passes, and is a dp on this.

A point through which *all* the curves of the family pass is a dp on Δ (§ 7). The directions of the tangents to the curves at this point may be a linear or a quadratic function of Ω In the first case every direction gives a different real value of Ω, therefore the curves pass singly in all directions through the fixed point. In order that they may do this the dp in Δ must have coincident tangents, for otherwise (since we cannot have real curves on both sides of any branch of Δ) certain directions would lie in excluded regions. On the other hand, when the directions are given by a quadratic function of Ω, some of the directions will determine imaginary values of Ω, therefore lie in excluded regions, and the curves will have contact in pairs.

See Ex. 14. The points $(1, 1)$, $(1, -1)$ are instances of fixed points of the latter kind.

The quartic may degenerate into

(1) cubic and line,

(2) two conics,

(3) conic and two distinct lines,

(4) conic and two coincident lines,

(5) four lines, two or three of which may be coincident.

(1) The line may be an envelope or a P.I.

(i) Suppose the line is an envelope; then the three points in which it meets the cubic are dps on Δ which are not fixed points of the family; for if a line cut every conic of the family at a given point it cannot also touch the conic, therefore it cannot be an envelope. Since then the points of intersection of the line and the cubic are dps on Δ which are not fixed points on the family, they are points through which one curve of the family passes twice. But, as has been stated above, the values of Ω giving these curves are repeated roots of the sextic which determines the line-pairs of the system, hence, in this case the sextic has *three repeated roots* and therefore the system contains only three line-pairs.

Ex. 15. $U = (x+1)^2\Omega^2 + 2y\Omega + y^2 - x = 0,$

$\Delta = x\{y^2(x+2) - (x+1)^2\}.$

Here the sextic is $(2\Omega^2 - 1)^2 = 0$, giving the three line-pairs $\Omega = \infty$, $\Omega = \pm 1/\sqrt{2}$.

(ii) Suppose the line is a P.I.; then P belongs to class 8 in the table (§ 17), and the equation of the family may be written

$$a'P\Omega^2 + 2b'P\Omega + c = 0,$$

where a', b' are lines and c a conic.

Then $$\Delta = P(a'c - b'^2P).$$

Here $P = 0$ passes through two fixed points of the family, viz. its intersections with c. Its remaining intersection with the cubic is its intersection with a', that is, the *dp* of the particular line pair $a'P$.

(2) When the quartic degenerates to two conics, both must be envelopes; for if either conic were a P.I., the family would have four fixed points (§ 16) and therefore be degenerate.

(3) When the quartic is made up of a conic and two different lines, these lines may be:—

(i) both envelopes,

(ii) one envelope and one P.I.,

(iii) both P.I.'s.

(4) If the two lines are coincident, they must be P.I.'s (table § 17).

(5) If $\Delta = 0$ split up into four lines these may be

(i) all different (α) all envelopes

(β) one envelope and three P.I.'s.;

(ii) two coincident and two distinct, the two coincident must be P.I.'s; the other two lines may be:—

(α) both envelopes,

(β) one envelope and one P.I.;

(iii) three coincident, all P.I.'s and one distinct, the envelope.

Some other special cases are mentioned in Salmon (*Higher Plane Curves*, Art. 273). If the family has two fixed points, k breaks up into the line joining them and a conic through them. The line is a P.I.; the conic is a proper tac-locus.

If a, b, c are all circles, the fixed points are the circular points; the envelope is then a bicircular quartic.

The Quadratic family of Cubics.

§ 21. This family has as envelope a sextic whose equation is

$$\Delta \equiv ac - b^2 = 0,$$

where a, b, c are cubics, and which therefore touches a and c at all their intersections with b. Every curve of the family has contact or quasi-contact with $\Delta = 0$ at the nine points of intersection of $a\Omega + b = 0$ and $b\Omega + c = 0$. The family cannot have more than seven fixed points, for if it have eight, it must also have another making nine and will therefore be a pencil of cubics and degenerate. It can, however, have a conic as part of two distinct or consecutive P.I.'s; for this necessitates only six fixed points, viz. the intersections of the conic with any cubic of the family.

The sextic may split up in various ways, as has already been shown in some of the examples; in this family node and cusp-loci first appear.

Ex. 16. $U \equiv x(x^2+y^2-2)\,\Omega^2 + 2y(x^2+y^2-2)\,\Omega + x(x^2-y^2) = 0.$ Fig. 8.

$$\Delta = (x^2+y^2-2)\,[x^4 - 2x^2y^2 - y^4 + 2y^2],$$

$$k = x(x-y)(x+y)(x^2+y^2-2),$$

$$\Delta_p = x^2(x-y)^2(x+y)^2(x^2+y^2-2)^3\,[x^4 - 2x^2y^2 - y^4 + 2y^2].$$

This is a family of cubics through seven fixed points.
The circle $x^2+y^2-2=0$ is a P.I. of type 8 (§ 17); $\mu=1$, $\nu=1$, $s=1$, $\rho=0$.

The Quadratic family of Quartics.

§ 22. In this family each curve has sixteen points of contact or quasi-contact with the octic $\Delta = ac - b^2 = 0$. A cubic may form part of two distinct or consecutive P.I.'s. The family cannot have more than twelve fixed points; for if it have thirteen, it must also have three more, viz. sixteen, and therefore will be a pencil.

Similarly we might proceed to discuss families of quintics, sextics, etc.

III. The Cubic Family.

Notation.

§ 23. The principal functions which present themselves in this family are (see Cayley, *Fifth Memoir upon Quantics.* Collected papers, No. 156, Vol. II.) as follows:—

(1) The general integral equation,

$$U = a\Omega^3 + 3b\Omega^2 + 3c\Omega + d = 0,$$

where a, b, c, d are algebraic functions of degree m in x, y.

(2) The Hessian, which is the quadratic family,

$$H = \mathrm{a}\Omega^2 + \mathrm{b}\Omega + \mathrm{c} = 0,$$

where we write a for $ac - b^2$,

b for $ad - bc$,

c for $bd - c^2$.

(3) The cubi-covariant, a cubic family of $3m$-ics;

$$\Phi = a'\Omega^3 + 3b'\Omega^2 + 3c'\Omega + d' = 0,$$

where

$$\begin{aligned} a' &= -a^2d + 3abc - 2b^3 = 2b\mathrm{a} - a\mathrm{b}, \\ b' &= -abd + 2ac^2 - b^2c = c\mathrm{a} - a\mathrm{c}, \\ c' &= \quad acd - 2b^2d + bc^2 = d\mathrm{a} - b\mathrm{c}, \\ d' &= \quad ad^2 - 3bcd + 2c^3 = d\mathrm{b} - 2c\mathrm{c}. \end{aligned}$$

(4) The Ω-discriminant of U, which is also the Ω-discriminant of H, and may be written

$$\Delta = 4\mathrm{ac} - \mathrm{b}^2.$$

(5) The first and second derived functions,

$$X = a\Omega^2 + 2b\Omega + c,$$

$$Y = b\Omega^2 + 2c\Omega + d,$$

$$X' = a\Omega + b, \quad Y' = b\Omega + c, \quad Z' = c\Omega + d,$$

and

$$X_H = 2\mathrm{a}\Omega + \mathrm{b}, \quad Y_H = \mathrm{b}\Omega + 2\mathrm{c}.$$

6. The p-equation,

$$V' = Ap^3 + 3Bp^2 + 3Cp + D = 0,$$

which is the eliminant of Ω between the equations

$$a\Omega^3 + 3b\Omega^2 + 3c\Omega + d = 0,$$

and

$$\frac{\mathrm{d}a}{\mathrm{d}x}\Omega^3 + 3\frac{\mathrm{d}b}{\mathrm{d}x}\Omega^2 + 3\frac{\mathrm{d}c}{\mathrm{d}x}\Omega + \frac{\mathrm{d}d}{\mathrm{d}x} = 0;$$

this eliminant is (see Salmon's *Higher Algebra*, p. 207)

$$-\left\{\left(a\frac{\mathrm{d}d}{\mathrm{d}x}-d\frac{\mathrm{d}a}{\mathrm{d}x}\right)-3\left(b\frac{\mathrm{d}c}{\mathrm{d}x}-c\frac{\mathrm{d}b}{\mathrm{d}x}\right)\right\}^3+27\,Q,$$

where
$$Q=\begin{vmatrix} \mathrm{a}, & \dfrac{\mathrm{da}}{\mathrm{d}x}, & \dfrac{\mathrm{d}a}{\mathrm{d}x}\dfrac{\mathrm{dc}}{\mathrm{d}x}-\left(\dfrac{\mathrm{d}b}{\mathrm{d}x}\right)^2 \\ \mathrm{b}, & \dfrac{\mathrm{db}}{\mathrm{d}x}, & \dfrac{\mathrm{d}a}{\mathrm{d}x}\dfrac{\mathrm{d}d}{\mathrm{d}x}-\dfrac{\mathrm{d}b}{\mathrm{d}x}\dfrac{\mathrm{dc}}{\mathrm{d}x} \\ c, & \dfrac{\mathrm{dc}}{\mathrm{d}x}, & \dfrac{\mathrm{d}b}{\mathrm{d}x}\dfrac{\mathrm{d}d}{\mathrm{d}x}-\left(\dfrac{\mathrm{dc}}{\mathrm{d}x}\right)^2 \end{vmatrix}.$$

Now $\dfrac{\mathrm{da}}{\mathrm{d}x}=\mathrm{d}_x+p\mathrm{a}$, etc.,

$$\frac{\mathrm{d}a}{\mathrm{d}x}\frac{\mathrm{dc}}{\mathrm{d}x}-\left(\frac{\mathrm{d}b}{\mathrm{d}x}\right)^2=(a_x+pa_y)(c_x+pc_y)-(b_x+pb\ )^2, \text{ etc.},$$

hence writing

$$\mathrm{a}_{xx}=a_xc_x-b_x^2,\quad \mathrm{a}_{xy}=a_xc_y+a_yc_x-2b_xb_y,\quad \mathrm{a}_y=a_yc\ -b\ ^2,$$
$$\mathrm{b}_{xx}=a_xd_x-b_xc_x,\quad \mathrm{b}_{xy}=a_xd_y+a_yd_x-b_xc_y-b_yc_x,\quad \mathrm{b}_{\ y}=a_yd_y-b_yc_y,$$
$$\mathrm{c}_{xx}=b_xd_x-c_x^2,\quad \mathrm{c}_{xy}=b_xd_y+b_yd_x-2c_xc_y,\quad \mathrm{c}_{yy}=b_yd_y-c_y^2,$$

we have

$$Q=\begin{vmatrix} \mathrm{a}, & \mathrm{a}_x+p\mathrm{a}_y, & \mathrm{a}_{xx}+p\mathrm{a}_{xy}+p^2\mathrm{a}_{yy} \\ \mathrm{b}, & \mathrm{b}_x+p\mathrm{b}_y, & \mathrm{b}_{xx}+p\mathrm{b}_{xy}+p^2\mathrm{b}_{yy} \\ \mathrm{c}, & \mathrm{c}_x+p\mathrm{c}_y, & \mathrm{c}_{xx}+p\mathrm{c}_{xy}+p^2\mathrm{c}_{yy} \end{vmatrix},$$

and by a simple transformation the p-equation becomes

$$-\tfrac{1}{27}\left[(ad_x)-3\,(bc_x)+p\,\{(ad_y)-3\,(bc_y)\}\right]^3$$
$$+\begin{vmatrix} \mathrm{a}, & \mathrm{a}_x, & \mathrm{a}_{xx} \\ \mathrm{b}, & \mathrm{b}_x, & \mathrm{b}_{xx} \\ \mathrm{c}, & \mathrm{c}_x, & \mathrm{c}_{xx} \end{vmatrix}+p\begin{vmatrix} \mathrm{a}, & \mathrm{a}_x, & \mathrm{a}_{xy} \\ \mathrm{b}, & \mathrm{b}_x, & \mathrm{b}_{xy} \\ \mathrm{c}, & \mathrm{c}_x, & \mathrm{c}_{xy} \end{vmatrix}+p\begin{vmatrix} \mathrm{a}, & \mathrm{a}_y, & \mathrm{a}_{xx} \\ \mathrm{b}, & \mathrm{b}_y, & \mathrm{b}_{xx} \\ \mathrm{c}, & \mathrm{c}_y, & \mathrm{c}_{xx} \end{vmatrix}$$
$$+p^2\begin{vmatrix} \mathrm{a}, & \mathrm{a}_y, & \mathrm{a}_{xy} \\ \mathrm{b}, & \mathrm{b}_y, & \mathrm{b}_{xy} \\ \mathrm{c}, & \mathrm{c}_y, & \mathrm{c}_{xy} \end{vmatrix}+p^2\begin{vmatrix} \mathrm{a}, & \mathrm{a}_x, & \mathrm{a}_{yy} \\ \mathrm{b}, & \mathrm{b}_x, & \mathrm{b}_{yy} \\ \mathrm{c}, & \mathrm{c}_x, & \mathrm{c}_{yy} \end{vmatrix}$$
$$+p^3\begin{vmatrix} \mathrm{a}, & \mathrm{a}_y, & \mathrm{a}_{yy} \\ \mathrm{b}, & \mathrm{b}_y, & \mathrm{b}_{yy} \\ \mathrm{c}, & \mathrm{c}_y, & \mathrm{c}_{yy} \end{vmatrix}=0.$$

Hence, multiplying by 27,

$$A = -[(ad\ ) - 3\,(bc_y)]^3 + 27 \begin{vmatrix} a, & a_y, & a_{yy} \\ b, & b_y, & b_{yy} \\ c, & c_y, & c_{yy} \end{vmatrix},$$

$$B = -[(ad_y) - 3\,(bc_y)]^2\,[(ad_x) - 3\,(bc_x)]$$
$$+ 9 \begin{vmatrix} a, & a_x, & a_{yy} \\ b, & b_x, & b_{yy} \\ c, & c_x, & c_{yy} \end{vmatrix} + 9 \begin{vmatrix} a, & a_y, & a_{xy} \\ b, & b_y, & b_{xy} \\ c, & c_y, & c_{xy} \end{vmatrix},$$

$$C = -[(ad_y) - 3\,(bc_y)]\,[(ad_x) - 3\,(bc_x)]^2$$
$$+ 9 \begin{vmatrix} a, & a_x, & a_{xy} \\ b, & b_x, & b_{xy} \\ c, & c_x, & c_{xy} \end{vmatrix} + 9 \begin{vmatrix} a, & a_y, & a_{xx} \\ b, & b_y, & b_{xx} \\ c, & c_y, & c_{xx} \end{vmatrix},$$

$$D = -[(ad_x) - 3\,(bc_x)]^3 + 27 \begin{vmatrix} a, & a_x, & a_{xx} \\ b, & b_x, & b_{xx} \\ c, & c_x, & c_{xx} \end{vmatrix}.$$

(7) If A, B, C, D have a common factor θ and

$$A = \theta\alpha,\ B = \theta\beta,\ C = \theta\gamma,\ D = \theta\delta,$$

then the "reduced p-equation" is

$$V = \alpha p^3 + 3\beta p^2 + 3\gamma p + \delta = 0.$$

If we write $\quad \mathfrak{A}$ for $\alpha\gamma - \beta^2$,

$\mathfrak{B}$,, $\alpha\delta - \beta\gamma$,

$\mathfrak{C}$,, $\beta\delta - \gamma^2$,

then the Hessian related to the p-equation is denoted by

$$H_p = \mathfrak{A}p^2 + \mathfrak{B}p + \mathfrak{C},$$

and $$\Delta_p = 4\mathfrak{A}\mathfrak{C} - \mathfrak{B}^2.$$

(8) From the fundamental relation, which in the case of the cubic family is

$$\Delta k^2 = \Delta_p \theta^4,$$

we can find k when Δ, Δ_p, and θ are known. Several different methods of finding k directly are, however, given

or stated by Casorati and Mr. Workman. A simple expression given by the latter may be quoted here. Let $\bar{\kappa}, \bar{\lambda}, \bar{\mu}, \bar{\nu}$ denote the minors of $\kappa, \lambda, \mu, \nu$ in the determinant

$$\begin{vmatrix} \kappa, & \lambda, & \mu, & \nu \\ a, & b, & c, & d \\ a_x, & b_x, & c_x, & d_x \\ a_y, & b_y, & c_y, & d_y \end{vmatrix};$$

and let δ denote the discriminant of

$$\bar{\kappa}\omega^3 + \bar{\lambda}\omega^2 + \bar{\mu}\omega + \bar{\nu} = 0.$$

Then $$k = a\frac{\partial\delta}{\partial\bar{\nu}} - b\frac{\partial\delta}{\partial\bar{\mu}} + c\frac{\partial\delta}{\partial\bar{\lambda}} - d\frac{\partial\delta}{\partial\bar{\kappa}}.$$

As already stated numerical factors are here disregarded.

(9) Another equation which may be mentioned here is the primitive of

$$\mathfrak{A}p^2 + \mathfrak{B}p + \mathfrak{C} = 0.$$

We may write this as

$$\mathfrak{H} = \mathfrak{a}\Omega^2 + 2\mathfrak{b}\Omega + \mathfrak{c} = 0.$$

There are then two quadratic families, H and $\mathfrak{H}$, of which the former has the same Δ as U and the latter has the same Δ_p. We see later that U, H and $\mathfrak{H}$ have the same envelope, etc.

Let m' be the degree of $\mathfrak{a}$ in x, y, and suppose the families have no singularities.

Then $\alpha = A$ and is of degree $3(2m-1)$, therefore $\mathfrak{A}$ is of degree $6(2m-1)$.

But referring to the quadratic family, we find that $\mathfrak{A}$ must be of degree $2(2m'-1)$; therefore

$$2(2m'-1) = 6(2m-1),$$

i.e., $$m' = 3m - 1.$$

General properties of a, b, c, &c.

§ 24. If a, b, c, d are curves of the m^{th} degree, a, b, c are of degree $2m$, and have at least $3m^2$ common intersections. For the identity

$$b\mathrm{b} \equiv c\mathrm{a} + a\mathrm{c}$$

shows that b, b form a degenerate $3m$-ic curve which passes, among others, through all the $4m^2$ intersections (ac). Now

b cannot meet c in more than m^2 of these intersections, for from the identity

$$\mathrm{c} \equiv bd - c^2$$

we see that b touches c at all the m^2 points (bc), making up the $2m^2$ intersections (cc) ; it follows then that b passes through all the $3m^2$ remaining intersections of a and c (cp. below the family of lines, $m = 1$, fig. 9). Let us call these common intersections of a, b and c the points L, M, N.

We have here shown that *the Hessian family is a quadratic family of curves of the* $2m^{\text{th}}$ *degree, having* $3m^2$ *fixed points* (L, M, N).

The points (L, M, N), being fixed points of the family, must be dps on $\Delta = 0$ (§ 7), and we can prove that they are *cusps*. For it is easily seen that

$$a^2\Delta \equiv -(a'^2 + 4\mathrm{a}^3).$$

which shows that $a^2\Delta$ has cusps on a at all the intersections $(a'\mathrm{a})$, see fig. 9. But remembering that

$$a' \equiv 2b\mathrm{a} - a\mathrm{b},$$

we see that the intersections $(a'\mathrm{a})$ are made up of the points $(a\mathrm{a})$ and the points (ab).

Now the points $(a\mathrm{a})$ are made up of m^2 contacts of a with a at (ab), and the points (ab) are made up of the intersections of $ac - b^2 = 0$ and $ad - bc = 0$, that is, of (ab), and the points (L, M, N), that is, the only points of the group $(a'\mathrm{a})$ which lie on a are the points (ab). Hence Δ *has cusps on* a *at all the remaining intersections of the group* $(a'\mathrm{a})$, *that is, at all the points* (L, M, N) ; *and the curve* a' *touches the cuspidal tangents at these points.*

Similarly from the identity

$$d^2\Delta \equiv -(d'^2 + 4c^3)$$

we see that d' touches all the cuspidal tangents at (L, M, N).

Again, since $\Delta = 4\mathrm{ac} - \mathrm{b}^2$, it has contact or quasi-contact with a and c at all their intersections with b. Now the intersections (ab) are made up of (L, M, N) and (ab), and a touches a at all the points (ab) ; therefore Δ touches a at all the points (ab), and similarly Δ touches c at all the points (cd). Collecting these results:—Δ *is a* $4m$-ic *having cusps at the* (m^2 *points* (L, M, N) *common to* a, b *and* c; *and moreover, touching a at the* m^2 *points* (ab), *and c at the* m^2 *points* (cd) ; *and lying in the regions where* a *and* c *have the same sign.* (Figs. 9, 11, &c.).

Note.—We use the symbols FU, FH, $F\Phi$ to denote "the family of curves represented by $U=0$, $H=0$, $\Phi=0$"; and we use H, Φ, etc. to denote any one of the curves belonging to the family $H=0$, $\Phi=0$, etc.

The relation of the curves of FU to the cusps on Δ must now be investigated. We know that three curves of FU pass through any point of the plane, and that if the point be chosen on Δ two of these curves are consecutive or coincident.

Note.—In the merely algebraic work the distinction between *consecutive* and *coincident* which is so important in the geometry of the subject is not explicitly shown.

Considering the question algebraically, we are given a cubic equation in Ω, $U=a\Omega^3+3b\Omega^2+3c\Omega+d=0$, whose coefficients are functions of x, y; at certain points of the plane, viz. on the curve $\Delta=0$, the values of x, y are such as to make two roots of this equation equal; we want to determine whether these roots are consecutive or coincident. Let us give to the point (x, y) a small arbitrary displacement in the plane, and so arrange that its position is determined by the value of one parameter ξ. Let the variation of this parameter be represented (fig. 10) by the displacement of a point along the line $O\xi$, and let the corresponding values of Ω be represented by the ordinates of the curve.

Thus, in general, all the roots are real and distinct, as at ξ_1; at ξ_2 two are coincident; beyond ξ_2 we have three real distinct roots again; at ξ_3 two are consecutive; beyond ξ_3 two are imaginary; at ξ_4 three roots are consecutive; on either side of ξ_4 two roots are imaginary; at ξ_5 two roots are consecutive and the third coincides with one of these; beyond ξ_5 the three roots are real and distinct; at ξ_6 the three roots are coincident, and beyond ξ_6 the three roots are still real; at ξ_7 two consecutive roots are coincident, and beyond ξ_7 two roots are imaginary, except at ξ_8 where there is an acnode. This makes up all the possible cases. The result, which is perfectly general is :—

On passing through two consecutive values the roots change from real to imaginary or imaginary to real;

in passing through two coincident values the roots change from $\begin{cases}\textit{real to real.}\\ \textit{imaginary to imaginary.}\end{cases}$

If the roots are *consecutive as well as coincident* they count as consecutive, see cusp at ξ_7 on the diagram.

Now suppose the part of Δ we are dealing with to be simply the envelope. Then through every point there pass two consecutive curves of FU, and one distinct from these. As we move along Δ suppose that at some point K the third curve actually coincides with one of the consecutive curves. Then at this point the equation $U=0$ has three equal roots in Ω. But the condition for this is $H=0$, $\Phi=0$, $\Delta=0$ (Cayley, *l.c.*, § 126), that is, $\mathrm{a}=0$, $\mathrm{b}=0$, $\mathrm{c}=0$ simultaneously. Therefore the point K considered is one of the cusps (L, M, N).

Again, the two coincident curves are met by the consecutive curve in m^2 points, and each of these points is, by the above, a cusp on Δ. Hence we have :—*the three values of Ω at a cusp on Δ are equal, and if a curve of the family $U=0$ passes through one cusp, it passes through m^2 cusps* (see below Ex. 23, § 36, fig. 21).

The Hessian Family.

§ 25. We now proceed to consider the Hessian family. We have already seen that it is a quadratic family of $2m$-ic curves having $3m^2$ fixed points and having the same Δ as FU.

Considered as a binary quantic in Ω, we know that

when $H=0$ has 2 real roots in Ω, $U=0$ has one real, two imaginary roots,
" 2 equal " " the same two equal roots,
" 2 imaginary " three real roots.

Now the roots of an equation change from real to imaginary and from imaginary to real in passing through two consecutive values, hence through any point there pass

$$\begin{cases} \text{two real } H\text{'s and one real } U, \\ \text{two consecutive } H\text{'s and the } \textit{corresponding} \text{ consecutive } U\text{'s}, \\ \text{no real } H\text{'s and three real } U\text{'s}, \end{cases}$$

according as the point is on the negative side of the envelope, on the envelope, or on the positive side.

Now on the first side there are two real curves of FH through any point, therefore these curves touch Δ on this side, but there is only one real curve of FU, therefore U touches Δ on the other side. The result then is:—*Corresponding curves of the cubic and Hessian families* (*i.e.*, *curves given by the same value of* Ω) *touch their envelope on opposite sides at the same point* (see point $P.$, fig. 12).

This result applies only at an ordinary point on Δ; at a dp there are two cases to be considered. For dealing with FH which is a quadratic family, we may have through a dp on Δ (see § 20)

(i) One of the curves of FH twice, or

(ii) An infinite number of curves of FH.

In case (i) $H=0$ has two equal roots at the dp, which must be *coincident*, therefore $U=0$ has the same two equal roots and these are also coincident (see Ex. 24, § 36). (This can be seen by considering the reality of the roots of $U=0$ and $H=0$ near the point). Hence one of the curves of FU passes twice through the point, that is, *two corresponding curves of FU and FH may have dps at a dp on Δ, and the branches may or may not touch the branches of Δ at the point.*

In (ii) a fixed ordinary point on FH is not a fixed point on FU; for suppose FU has a fixed point and let it be taken as origin. Then the terms of lowest degree in a, b, c, d must be linear in x, y, that is:—*The Hessian family must have a fixed dp at a fixed ordinary point on the cubic family.*

It follows then that a fixed ordinary point on FH is not a fixed point on FU. At such a point, however, a, b, c all vanish, therefore H vanishes identically, and this is the condition (Cayley, *l.c.*, § 126) to be satisfied in order that $U=0$ may have three equal roots in Ω, that is, the three U curves through the point are the same.

Again, *the Hessian has a crunode where the corresponding curve of the cubic family has an acnode, and conversely.*

As we cross the node-locus, the roots of $\left\{\begin{matrix}H\\U\end{matrix}\right.$ change from $\left\{\begin{matrix}\text{2 real and different}\\\text{1 real and 2 imaginary}\end{matrix}\right.$ through $\left\{\begin{matrix}\text{2 coincident}\\\text{2 coincident, 3 real}\end{matrix}\right.$ back to $\left\{\begin{matrix}\text{2 real and different}\\\text{1 real, 2 imaginary}\end{matrix}\right.$; or from $\left\{\begin{matrix}\text{2 imaginary}\\\text{3 real}\end{matrix}\right.$ through $\left\{\begin{matrix}\text{2 coincident}\\\text{2 coincident, 3 real}\end{matrix}\right.$ back to $\left\{\begin{matrix}\text{2 imaginary}\\\text{3 real}\end{matrix}\right.$.

It is evident that a change from 2 real roots through 2 coincident back to 2 real necessitates a crunode, and the similar change for imaginary roots an acnode.

The Hessian and the corresponding curve of the cubic family have cusps at the same point, having the same tangents but turned opposite ways.

By considering a cusp as an evanescent loop or as the result of the coalescence of an acnode with a branch of the curve, this theorem follows from the two proved above.

Similarly when a curve of FU has a singularity consisting in its penultimate form of a series of λ nodes, the corresponding curve of FH has a singularity of the same type at the same point, what are crunodes on one being acnodes on the other. (Whether a singularity is accurately described as a series of acnodes is a question to be determined by Klein's equation; but this abbreviated description is here adopted for simplicity).

If the singularity on the curve of FU has as its penultimate form a loop followed by λ nodes, the corresponding curve of FH has a singularity of the same type turned in the opposite direction. The singularities here dealt with are the only ones which can occur on FH, which is a quadratic family (see § 6).

When two consecutive or coincident curves of FU have one or more repeated branches in common the corresponding consecutive or coincident curves of FH have the same repeated branches in common.

Suppose Q is the repeated branch considered, then, using the method of § 15, we can, by a linear transformation of Ω, bring the equation U to the form

$$a' Q^\mu \Omega^3 + 3b' Q^\nu \Omega^2 + 3c\Omega + d = 0,$$

where a', b', c, d, do not contain Q as a factor.

Then $\mathrm{a} = a'cQ^\mu - b'^2 Q^{2\nu}$, $\mathrm{b} = a'dQ^\mu - b'cQ^\nu$, $\mathrm{c} = b'dQ^\nu - c^2$.

Now if $\mu = 2\nu + r$, $r > 0$, $\mathrm{a} = Q^{2\nu}(a'cQ^r - b'^2)$,

$$\mathrm{b} = Q^\nu (a'dQ^{\nu+r} - b'c),$$

$$\mathrm{c} = \quad (b'dQ^\nu - c^2),$$

and Q is a P.I. of $H = 0$ of type 7, in table, § 17.

If $\mu = 2\nu - s$, $s > 0$, $\mathrm{a} = Q^{2\nu - s}(a'c - b'^2 Q^s)$,

$$\mathrm{b} = Q^\nu (a'dQ^{\nu - s} - b'c),$$

$$\mathrm{c} = b'dQ^\nu - c^2; \text{ provided } \nu \mathrel{\substack{=\\>}} s;$$

but if $\nu < s$, $= s - t$ say, $\mathrm{a} = Q^{s-2t}(a'c - b'^2 Q^s)$,

$$\mathrm{b} = Q^{s-2t}(a'd - b'cQ^t),$$

$$\mathrm{c} = \quad b'dQ^{s-t} - c^2.$$

In both these cases Q is of type 8 (§ 17).

If $\mu = 2\nu$, $\mathrm{a} = Q^{2\nu}(a'c - b'^2)$,

$$\mathrm{b} = Q^\nu (a'dQ^\nu - b'c),$$

$$\mathrm{c} = \quad b'dQ^\nu - c^2.$$

Here Q is of type 7 (§ 17).

This shows the correspondence between certain P.I.'s of FU and FH, namely, those at every point of which the equation $U = 0$ has two equal roots in Ω. The number of occurrences of Q in Δ can be found from the table (§ 17).

Determination of the Geometrical functions of factors of Δ.

§ 26. We have now shown that, *when* a, b, c *have no common factor*

the discriminants of FU and FH are the same,
the envelope of FU is the envelope of FH,
the node-locus of FU is the node-locus of FH,
the cusp-locus of FU is the cusp-locus of FH,
the locus of a certain higher singularity on FH is the locus of a similar singularity on FU,
a P.I. forming part of two consecutive or coincident curves of FU is a P.I. forming part of two consecutive or coincident curves of FH.

Hence we have reduced the problem of finding the geometrical significance of any factor of Δ for a cubic family to the same question for a quadratic family, and this has already

been completely solved (§ 17). All then that we require to know is the p-discriminant and the k and θ for FH, which we may denote by $\overline{\Delta}_p(H)$, $\overline{k}(H)$, and $\overline{\theta}(H)$, and which are easily found.

Note—Since on the envelope, node-locus, cusp-locus, etc., of FU it is only necessary that a certain condition be satisfied for *two* of the values of Ω and *two* of the values of p at every point, the third Ω and p are unconditioned, and it may happen that this Ω is the same at every point of the locus, that is, *the envelope, node-locus, cusp-locus, or singularity locus of FU may also be a particular integral.* This, however, does not affect the occurrence of the corresponding factors in FU or their relation to FH. Keeping strictly to the definition the envelope in this case would not be a singular solution. Boole describes it as "a particular integral possessing the geometrical characters of a singular solution," (*Differential Equations*, 3rd edition, p. 172), but Hamburger (No. 27, p. 218) admits that a solution can be both singular and particular. Boole gives the following example of an envelope which is also a P.I.

$$U = \Omega^3 - 6x\Omega^2 + 9x^2\Omega - y = 0,$$

$$H = -x^2\Omega^2 - (y - 6x^3)\,\Omega + 2xy - 9x^4 = 0,$$

$$\Delta = -y\,(y - 8x^3)\,;$$

$y = 0$ is a P.I. and also a branch of the envelope.

Ex. 17. In this example $y = 0$ is a node-locus and also a P.I.

$$U = \Omega^3 - 6x\Omega^2 + 9x^2\Omega - xy^2 = 0. \quad \text{Fig. 12},$$

$$H = -x^2\Omega^2 - x\,(y^2 + 6x^2)\,\Omega + x^2\,(2y^2 - 9x^2) = 0,$$

$$\Delta = x^2y^2\,(y^2 - 4x^2)\,;$$

$y = 0$ is a triple-point locus for $F\Phi$;

$$\Phi = (\Omega - 3x)^2\,\{\Omega y^2 - 2x^2\,(\Omega - 3x)\} + y^2\,\{6x^2\,(\Omega - 3x) + xy^2\} = 0.$$

If, however, a, b, *and* c *have a common factor*, this (see Burnside and Panton, *Theory of Equations*, p. 340) must be the locus of points at which the equation $U = 0$ has three equal roots in Ω. Leaving for the moment the question of P.I.'s, these roots may be

(i) *Three coincident*, that is, the *same* curve passes three times through the point, and the locus is a *locus of triple points.* The number of real roots of the equation is not changed by a passage through three coincident roots (provided they are not also consecutive, § 24, note), hence the locus must occur raised to an even power in Δ (see Φ in Ex. 17). This applies when the tangents at the triple point are all distinct; but if *two* of the tangents coincide, then an additional intersection of every pair of consecutive curves has moved up to the singularity, and a branch of the envelope has coincided with the singularity locus, giving an additional factor in Δ, and raising the locus of triple points to an odd power in Δ.

In this case two of the roots are consecutive *as well as* coincident. If the *three* tangents are coincident, a second intersection of pairs of consecutive curves has moved up to the triple point, the singularity locus occurs to an even power in Δ, and the three roots are consecutive as well as coincident.

An example of the locus of a triple point composed of a cusp with a branch through it is $x = 0$ in

Ex. 18. $U = \Omega^3 + 3(x+y)\Omega^2 + 3(2xy+y^2)\Omega + y^3 + 3xy^2 - x^4 = 0$. Fig. 13,

$$H = x^2[\Omega^2 + (2y+x^2)\Omega + y^2 + x^2y + x^3] = 0,$$

$$\Delta = x^7(x-4).$$

It will be noticed that corresponding curves of FH and FU have cusps at the same point turned opposite ways. The family is formed by moving the curve U_0 parallel to itself in the direction of the y-axis.

A locus of triple points with coincident tangents of the type $y^3 = x^4$ is shown in

Ex. 19. $U = x^3\Omega^3 - 3x^2y\Omega^2 + 3xy^2\Omega - y^3 + (x+y^2)^4 = 0$. Fig. 14,

$$H = x^2(x+y^2)^4[x\Omega - y] = 0,$$

$$\Delta = x^6(x+y^2)^8.$$

The locus of the singularity, which is the same on $F\Phi$, is $x + y^2 = 0$.

$$\Phi = x^2(x+y^2)^4[(x\Omega - y)^3 - (x+y^2)^4] = 0,$$

therefore $\Phi = 0$ has a singularity of the same type as that of the corresponding U and at the same point.

(ii) The roots may be *two coincident and the third consecutive to these*, that is, a curve passes twice through the point, and the consecutive curve passes once, so that the *locus is both node-locus and envelope* (fig. 15, A). On passing through a point where two roots are coincident and one consecutive to these, the number of real roots changes (note, § 24), therefore the locus is raised to an odd power, in general the third, in Δ. It will be seen later that every cubicovariant family gives an example of this. The two tangents at the node are in general distinct; if they coincide, so that the dp becomes a cusp, then a locus of intersections of pairs of consecutive curves coalesces with the node-locus, giving this locus raised to an even power (in general the fourth) in Δ. In this case the nodes have become cusps, and we have *a cusp-locus, the tangent at each cusp being a tangent to the locus at the point* (fig. 15, B).

(iii) *The three roots may be consecutive*, that is, three consecutive curves pass through a point. In order that this may happen, two consecutive curves must have two consecutive common points, and *each curve will then be found to have three-point contact with its envelope* (fig. 16). This locus does not appear to have been clearly recognised in the Theory of

Singular Solutions*: it is quite different in nature from the combination of two envelopes (which combination cannot occur in the cubic family), and might perhaps be called the *osc-envelope*. It occurs squared in both Δ and Δ_p.

Ex. 20. $U = x^2y\Omega^3 + 3xy\Omega^2 + 3y\Omega + 1 = 0$. Fig. 17,

$$H = y(x-y)[x\Omega + 1],$$

$$\Delta = x^2y^2(x-y)^2.$$

Here $x - y = 0$ is an osc-envelope, $x = 0$ and $y = 0$ are P.I.'s.

$$\Phi = y(x-y)[y(x\Omega + 1)^3 - x + y] = 0,$$

hence $x - y = 0$ is also an osc-envelope for $\Phi = 0$.

As regards P.I.'s which are the loci of points at which the equation $U = 0$ has three equal roots in Ω, no classification is attempted. The most general equation of this type which has $R = 0$ as a P.I. is

$$a'R^\mu\Omega^3 + 3b'R^\nu\Omega^2 + 3c'R^\rho\Omega + d = 0,$$

(where a', b', c', d are supposed not to contain R as a factor), and in this

$$\mathrm{a} = a'c'R^{\mu+\rho} - b'^2 R^{2\nu},$$

$$\mathrm{b} = a'dR^\mu - b'c'R^{\nu+\rho},$$

$$\mathrm{c} = b'dR^\nu - c'^2 R^{2\rho},$$

that is, R is a common factor of a, b, c.

Hence a common factor of a, b, c is either

(i) a locus of triple points, tangents all different, two coincident or three coincident;

(ii) locus of nodes with one branch touching the node-locus, or locus of cusps with the cuspidal tangent touching the locus;

(iii) locus of two-point contacts of consecutive curves (osc-envelope);

(iv) particular integral.

The above is an enumeration of all the possible functions of factors of Δ.

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* Boole remarks (No. 1, p. 181) that the contact of the curve with the singular solution is "not generally of the second order." De Morgan gives an example of a family of curves having three-point contact with their envelope.

The Cubicovariant Family.

§ 27. We next proceed to consider

$$\Phi = a'\Omega^3 + \ldots = 0.$$

It is important to notice that the discriminant of Φ is Δ^3. We have already seen (§ 24) that a' and d' touch the cuspidal tangents of Δ at all the points (L, M, N). From the identity

$$b'^2 + 4a^2c = -b^2\Delta,$$

it follows that b' also touches these tangents; for b' passes through all the intersections (ac), and therefore through those considered, viz. (L, M, N) which do not lie on b. Similarly, c' has contact with Δ at the cusps. Hence *the cuspidal tangents of Δ are tangents at the points (L, M, N) to all the curves of the family Φ, that is, $F\Phi$ has $3m^2$ fixed tangents* (fig. 11).

Again writing $\overline{H}$ to denote the operation of taking the Hessian (Cayley, *l.c.*, § 117), we have

$$\overline{H}\Phi = -\Delta H.$$

Now we have already proved that corresponding curves of the cubic and Hessian families touch their envelope at the same point on opposite sides, therefore corresponding curves of $F\Phi$ and $F\overline{H}\Phi$, that is, of $F\Phi$ and FH touch their envelope on opposite sides at the same point. But corresponding curves of FU and FH also touch the envelope on opposite sides at the same point; hence *corresponding curves of FU and $F\Phi$ touch their envelope on the same side at the same point* (see point P, fig. 11).

When Δ denotes only the envelope, this accounts for one of the factors Δ in $\overline{\Delta\Phi}$. The remaining Δ^2 is due to the fact that Δ is also a node-locus for Φ. To shew that this is the case, consider the curve U for $\Omega = \infty$; take one of its points of contact with its envelope for origin, and take the tangent there for axis of x. Then

$$a = a_2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \ldots,$$

and since the consecutive curve passes through the origin

$$b = b_1x + b_2y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \ldots,$$

and
$$c = c_0 + c_1x + c_2y + \ldots,$$

$$d = d_0 + d_1x + d_2y + \ldots.$$

Hence

$$\mathrm{a} = a_2 c_0 y + (a_{20} c_0 - b_1^2) x^2 + \ldots,$$

$$\mathrm{b} = - b_1 c_0 x + (a_2 d_0 - b_2 c_0) y + (a_{20} d_0 - b_1 c_1 - b_{20} c_0) x^2 + \ldots,$$

$$\mathrm{c} = - c_0^2 + (b_1 d_0 - 2c_0 c_1) x + (b_2 d_0 - 2c_0 c_2) y + \ldots;$$

and

$$a = 2b\mathrm{a} - a\mathrm{b} = a_2 y \{3c_0 b_1 x + (3c_0 b_2 - a_2 d_0) y\} + b_1 (3a_{20} c_0 - 2b_1^2) x^3 + \ldots.$$

Therefore the origin is a dp on Φ for $\Omega = \infty$, having $y = 0$ as one of its tangents. The general statement of this is:— *Every curve of $F\Phi$ has a node on the envelope with one branch touching the envelope, and touching the corresponding curves of FU and FH; the curves Φ and U lie on one side, and H on the opposite side of the envelope* (fig. 11, P; fig. 12, P, Q, &c.).

Again, at the origin the shapes of $a = 0$, $a' = 0$, $\Delta = 0$, and $\mathrm{a} = 0$ are given by

$$y + \frac{a_{20}}{a_2} x^2 = 0,$$

$$y + \left(\frac{a_{20}}{a_2} - \frac{2}{3} \frac{b_1^2}{a_2 c_0}\right) x^2 = 0,$$

$$y + \left(\frac{a_{20}}{a_2} - \frac{3}{4} \frac{b_1^2}{a_2 c_0}\right) x^2 = 0,$$

$$y + \left(\frac{a_{20}}{a_2} - \frac{b_1^2}{a_2 c_0}\right) x^2 = 0.$$

Hence *the curves lie in the order U, Φ, Δ, H* (figs. 11, 12, &c.)

The singularity (not accidental) on $F\Phi$ which corresponds to a node (not accidental) on FU is evidently a triple point. For if $\xi = 0$ is the node-locus of FU, then $\overline{\Delta\Phi}$ has ξ^6 as a factor. It will be seen, by drawing the different penultimate forms, that a crunode on FU corresponds to a triple point with three real branches on $F\Phi$; and an acnode on FU to a triple point with only one real branch (figs. 18, A and B).

Again, if $\eta = 0$ is a cusp-locus of FU, then η^9 is a factor of $\overline{\Delta\Phi}$. It will be found that the singularity on $F\Phi$ is a cusp with a branch through it touching the cuspidal tangent (figs. 18, C and D).

The singularities on $F\Phi$ that correspond to higher singularities on FU can be easily found in any special case. The

investigation of the general case, as also that of the singularity corresponding to any fixed or accidental singularity on FU, is not of much interest.

The derived functions.

§ 28. Since Δ is the eliminant of $X=0$ and $Y=0$, all the intersections of corresponding curves of FX and FY lie on $\Delta=0$. Points of ordinary intersection have a simple factor for their locus and form the envelope; points of ordinary contact give a squared factor in Δ, the node locus of FU; the locus of three-point contact of $X=0$ and $Y=0$ is the cusp-locus of FU; and so on.

Similarly, corresponding curves of FX_H and FY_H intersect on the envelope of FH, touch on the node-locus of FH, and have three-point contact on the cusp-locus of FH, etc. But corresponding curves of FU and FH touch their envelope at the same point, have their nodes at the same point, have their cusps at the same point, etc.; and

$$U=\Omega X+Y,\ 2H=\Omega X_H+Y_H.$$

Hence at any point where a corresponding X and Y intersect, the corresponding X_H and Y_H intersect; where the corresponding X and Y touch, the corresponding X_H and Y_H touch; and so on.

Again, we have the expression for Φ,

$$\Phi=XY_H-X_HY;$$

and this shows, from what has just been proved, that a point where U touches its envelope is at least a dp on the corresponding Φ, as we have already stated.

The p-equation.

§ 29. Throughout this discussion, which has dealt with the geometrical significance of the factors of Δ, no use has been made of the cubic p-equation, $V=0$, derived from the family $U=0$ (§ 23, 6). It has been shown that the factors of Δ, with some easily recognised exceptions, have the same functions in the family H as in the family U, and these functions can therefore, by our previous work in the quadratic family (§ 17), be at once determined from the p-equation of $H=0$, without referring to the cubic p-equation, $V=0$.

Now, starting with the p-equation

$$V = \alpha p^3 + 3\beta p^2 + 3\gamma p + \delta = 0$$

and its Hessian

$$H_p = \mathfrak{A}p^2 + \mathfrak{B}p + \mathfrak{C} = 0,$$

and considering p for the moment as an arbitrary parameter, the equations $V=0$, $H_p = 0$ are of exactly the same nature as $U=0$, $H=0$ ($V=0$ is in fact the locus of contacts of parallel tangents to $U=0$, see Professor Hill's paper, No. 22); therefore the factors of Δ_p have the same functions in H_p as in V, and, since the arguments depend on analytical relations only, this still holds when $V=0$ and $H_p=0$ are ordinary p-equations. Hence, if we find the primitive of $H_p = 0$ (§ 23, 9),

$$\mathfrak{H} \equiv \mathfrak{a}\Omega^2 + 2\mathfrak{b}\Omega + \mathfrak{c}$$

(which is not the Hessian of the Ω-equation), we can by means of this determine the functions of the factors of Δ_p.

Hence, *given a cubic integral equation or a cubic p-equation, its geometrical properties can be explained by means of a quadratic integral equation and a quadratic p-equation.*

The Cubic family of Lines.

§ 30. This family, which is the special case $m=1$, gives interesting illustrations of the preceding results, some of which can be proved more easily than in the general case.

It is at once evident that the envelope is a unicursal quartic of the third class, that is, *the discriminant is a tricuspidal quartic.*

The relations of a, b *and* c (figs. 9, 19).

a, $\equiv ac - b^2$, is a conic touching a, c at the points (ab), (bc),

c, $\equiv bd - c^2$, „ „ b, d „ (bc), (cd).

Hence the conics a and c intersect at (bc), and at three remaining points which we have called (L, M, N).

b, $\equiv ad - bc$, is a conic through the four points (ab), (bd), (ac), (cd); and from the identity

$$b\mathrm{b} \equiv \mathrm{c}a + a\mathrm{c}$$

we see that it passes through the points (L, M, N). Hence (L, M, N) are the three common points of a, b, c, *i.e.*, are fixed points of the Hessian family, and from what we have seen above (§ 24) are the cusps of the tricuspidal quartic.

The conics a, b and c are then specially related to the complete quadrilateral formed by *a*, *b*, *c*, *d*. If a, b and c are chosen arbitrarily, no corresponding cubic family can, in general, be found; but if only two of the conics are given, the family (*i.e.*, *a*, *b*, *c*, *d*) can be determined by a linear construction, though not uniquely. We have, in fact:—

Given the conics a, c, *four families are determined.* For, at any one of the four intersections of a and c, draw the tangents *b* and *c* to c and a respectively (fig. 9).

Where *b* meets a again draw the tangent *a*,

" c " c " " *d*;

hence *a*, *b*, *c*, *d* are found, and since we might have started with any other intersection of a and c, *four* different families are determined.

There is an exception to this when one of the conics is a line-pair. Suppose c is

(i) A line-pair with its dp not on a. In this case we must take one of the lines as *b*; the tangents where it meets a are *a* and *c*. It is now impossible to determine *d* so that c may be $\equiv bd - c^2 = 0$.

(ii) A line-pair with its dp on a. Here one construction is possible; take the tangent to a at the dp as *c*, and take its harmonic conjugate with respect to the line-pair as *b*. This determines *a*, the tangent at the remaining intersection of *b* and a, and since $\mathrm{c} = \lambda b^2 - c^2$, where λ is a constant, $d = \lambda b$, therefore the family is

$$a\Omega^3 + 3b\Omega^2 + 3c\Omega + \lambda b = 0.$$

Given the conics a, b, *four quadrilaterals are determined.* Choose any intersection of a and b, and at this point draw the line *a* touching a. This meets b again in (*ac*). From this point draw the tangent *c* to a. The join of the points of contact of the two tangents from (*ac*) to a is *b*; draw *b* meeting b in (*bd*). Then, since *a* meets b in (*cd*), *d* is found by joining (*bd*), (*cd*). Hence the cubic family is determined when one of the points of intersection of a and b is chosen, therefore given a and b four families can be constructed.

This construction holds if b is a line-pair, but fails if a is a line-pair.

Given a, b, c, d, every line of the family can be found by a linear construction (fig. 9).

To show this, we need the relations

$$X = \Omega X' + Y', \quad Y = \Omega Y' + Z', \text{ and } U = \Omega X + Y.$$

Since $X' = a\Omega + b$, $Y' = b\Omega + c$, we see that

X' and Y' intersect on a and X is tangent to a at $(X'Y')$,

Y' „ Z' „ c „ Y „ c „ $(Y'Z')$.

Now, given a, b, c, d, we can at once draw a and c, and these, with Y', determine X' and Z'. Hence the construction is:—draw any line, Y', through (bc); the tangents to a and c at its other points of intersection with them are a corresponding X and Y.

Now $\qquad U = \Omega X + Y,$

and

$$\begin{aligned} -\Omega X + Y &= -a\Omega^3 - 2b\Omega^2 - c\Omega \\ &\quad + b\Omega^2 + 2c\Omega + d \\ &= -X'\Omega^2 + Z'. \end{aligned}$$

Hence U is the harmonic conjugate with respect to X, Y of a line joining (XY) and $(X'Z')$, and can be found by a linear construction. This construction fails when Y' passes through a common point of a and c, that is, at a cusp on Δ.

The Discriminant.

§ 31. We have here found a linear construction for $\Delta = 0$, the tricuspidal quartic which is the envelope of U. The different forms of Δ are shown in figs. 9, 19, and 20. Since it is the locus of intersections of corresponding tangents of a and c it lies entirely outside both these conics. There is only one way in which it can degenerate, viz. into a cuspidal cubic and a line (fig. 20), for any other kind of degeneration would alter the class. In this case two of the points L, M come together, and the line must be the inflexional tangent to the cubic at this point. The line, $P = 0$, say, is a P.I., and by a linear transformation, such as that used above in the quadratic family, we can make it the P.I. for the value $\Omega = \infty$.

The equation then becomes

$$P\Omega^3 + 3\lambda P\Omega^2 + 3c\Omega + d = 0$$

where λ is any constant.

We have

$$\mathrm{a} = P(c - \lambda^2 P), \quad \mathrm{b} = P(d - \lambda c), \quad \mathrm{c} = \lambda P d - c^2;$$

$$\Delta = P\{P(6\lambda cd - 4\lambda^3 dP + 3\lambda^2 c^2 - d^2) - 4c^3\};$$

showing that P is the inflexional tangent at (Pc) to the cubic. A similar result holds if one of b, c is absent.

Ex. 21. $U \;= x\Omega^3 + 3y\Omega + x + 2y + 1 = 0$. Fig. 20,

$H \;= xy\Omega^2 + x(x + 2y - 1)\,\Omega - y^2 = 0,$

$\Delta \;= x(x^3 + 4x^2y + 4xy^2 + 4y^3 - 2x^2 - 4xy + x),$

$\Phi_{-1} = 2x^3 + 3x^2y + 4xy^2 + 2y^3 - 3x^2 - xy + x = 0.$

The discriminant has split up into a cuspidal cubic and the P.T. $x = 0$. The two points L, M have come together at the origin and at an inflexion on the envelope.

The Hessian Family.

§ 32. This is a quadratic family of conics with three fixed points L, M, N. It has therefore contact with Δ at only one point, namely the point at which the corresponding U touches Δ.

The equation $H = 0$ may be written

$$(a\Omega + b)(c\Omega + d) = (b\Omega + c)^2,$$

i.e. $$X'Z' = Y'^2;$$

which shows that H touches X', Z' at the points $(X'Y')$, $(Y'Z')$.

At a cusp on Δ, L, say, X', Y', Z' pass through one point, viz. L, and therefore H degenerates to a line-pair with its dp at L. But H always passes through M and N, therefore in this case consists of the lines LM and LN.

The Cubicovariant Family.

§ 33. This is a cubic family of cubics with three fixed tangents, for every curve touches the cuspidal tangents of Δ at L, M, N. This accounts for nine of the intersections of the cubic and the quartic. The remaining three come together at a node on Φ, with one branch touching Δ where the corresponding U and H curves touch it (P, fig. 11), *e.g.* the curve a' given by $\Omega = \infty$ touches a at (ab) and has a dp there; it passes through L, M, N and touches the cuspidal tangents at these points; (bd), another intersection of b and b, is another known point on a', that is we know enough to completely determine the curve, provided the tangents at L, M, N are known. These tangents are concurrent, therefore two suffice to determine the third. Similarly for d'.

It has already been pointed out that a combination of envelope and node-locus cannot occur in a quadratic family, hence, though in general an envelope of $F\Phi$ corresponds to an envelope of FH and a node-locus of $F\Phi$ to a node-locus of FH, yet in the special case in which the envelope and node-locus of $F\Phi$ coincide, their locus, Δ, divides out from the equation of the Hessian of $F\Phi$, that is,

$$\overline{H}\Phi = -\Delta H.$$

There is a similar reduction in H for the case of other combinations of loci and for loci of higher singularities, but this particular reduction occurs in every cubicovariant family.

We have proved that Φ touches each cuspidal tangent of Δ and has a dp at the point of contact of U and Δ with U as tangent, therefore, as U moves up towards the cusp the dp approaches the cusp, and the two tangents at the dp tend to coincide in direction. Ultimately, when U becomes the cuspidal tangent, the dp on Φ has become a cusp with U as tangent (Fig. 20). This is easily seen from a diagram.

Ex. 22. $U = (x - y + 1)\,\Omega^3 + 6x\Omega^2 + 6\,(x + y)\,\Omega + 3x + 4y - 1 = 0.$ Fig. 11,

$$H = -2\,(x^2 + y^2 - x - y)\,\Omega^2 - (x^2 + 3xy + 4y^2 - 2x - 5y + 1)\,\Omega + 2\,(x^2 - 2y^2 - x) = 0,$$

$$\Delta = -16\,(x^2 + y^2 - x - y)\,(x^2 - 2y^2 - x) - (x^2 + 3xy + 4y^2 - 2x - 5y + 1)^2$$

$$a' = \Phi_\infty = -7x^3 + 2x^2y - 7xy^2 - 4y^3 + 7x^2 + 8xy + 9y^2 - x - 6y + 1 = 0,$$

Φ_∞ has a node at (ab), the point $(0, 1)$ which is the point of contact of U_∞ and H_∞, and one branch touches H_∞ (Point P, Fig. 11). Φ also touches the cuspidal tangents of Δ at L, M, N.

The Cubic family of Conics.

§ 34. This is most easily investigated by referring to the cubic family of lines. The arrangement of special points in the family of lines shows the position and relation of groups of points in the family of conics.

Relations of a, b, c.

Since a, b, c, d are conics, a, b, c are quartics, therefore a, c have 16 points in common; b passes through all these points except the group (bc), that is, a, b, c have 12 common points, and therefore the Hessian family has 12 fixed points, which we take in groups of four and call (L, M, N).

This follows from the relation

$$b\mathrm{b} = c\mathrm{a} + a\mathrm{c},$$

which shows that b, b form a degenerate sextic through

16 points (ac), made up of 4 points (bc), and 12 points (L, M, N);

8 „ (aa), „ 4 „ (ab), contacts of a, a;

8 „ (cc), „ 4 „ (bc), „ b, c,

and of 4 points (cd), contacts of d, c;

4 „ (ac).

Of these points only (ab) and (bc) lie on b, hence b passes through all the remaining ones, and from its equation we know it passes through the 16 points (ab), (ac), (bd) and (cd); that is, b passes through the 12 points (L, M, N) and the 16 points (ab), (ac), (bd) and (cd)—28 known points.

In this family it is not in general possible, as it was for the family of lines, to find a, b, c, d, given a and c. For an extension of the construction given in that case requires us to draw a conic, b, touching a quartic, c, in four points, which cannot in general be done. Similarly, given a and b, no construction is possible.

The Discriminant (figs. 21, 22).

§ 35. This is seen from its equation, $\Delta = 4\mathrm{ac} - \mathrm{b}^2 = 0$ to be an 8-ic, touching a at four intersections of a and b, and touching c at four intersections of b and c. It has 12 cusps at the points (L, M, N) common to a, b and c. Every conic of the family touches Δ at the four intersections of the corresponding X and Y. This accounts for 8 out of the 16 intersections of the conic and Δ, therefore every conic must cut Δ at 8 points.

The Hessian Family.

§ 36. Supposing for the present that FU has no singularities or combinations of loci that cannot occur in a quadratic family, and therefore that a, b and c have no common factor, the Hessian is a quadratic family of quartics, having the 12 points (L, M, N) which are common to a, b, and c, as fixed points, and therefore in general having no other fixed point. Each curve has 16 points of contact or quasi-contact with Δ; 12 of these points being at the cusps, the remaining 4 being the 4 points of contact of the corresponding curve of FU with Δ.

Again, since the equation of H may be written

$$X'Z' = Y'^2,$$

H touches X' and Z' at all the points where Y' meets them, *i.e.* at four points each.

The curve of the Hessian Family at a cusp.

We have already shown (§ 24) that if a particular curve $U_\omega = 0$ passes through one cusp it will pass through four, see U_1 in fig. 21, and at each of these points $U = 0$, considered as an equation in Ω, will be a perfect cube $(\Omega - \omega)^3$. At any one of these four cusps then X and Y will be perfect squares in $\Omega - \omega$, that is, the X and Y corresponding to U intersect at the four cusps through which U passes. Again, since X and Y are perfect squares in $\Omega - \omega$ at each cusp, the X', Y', and Z' at the cusp will be given by the same value $\Omega = \omega$, and will intersect at each of the cusps; therefore the particular X', Y', and Z' corresponding to any particular cusp have four points in common, viz. the four cusps of Δ which lie on the U determined by the cusp chosen.

Returning now to the equation of the Hessian family,

$$X'Z' = Y'^2,$$

we see that these four common points of X', Y', and Z' must be dps on the corresponding curve of FH, and this, being a quartic with four dps, must degenerate to (i) a line and a nodal cubic, or (ii) two conics.

In case (i) three of the dps are on a line and this line must therefore form part of each of the conics X', Y', Z' therefore part of the corresponding U, that is, the line is part of a P.I. Our result then is:—

The particular curves of FU and FH which correspond to a particular cusp on Δ also pass through three other cusps, and the quartic curve, $H = 0$, has dps at each of the cusps and therefore degenerates either into two conics, or into a nodal cubic and a line. In the latter case the line is a P.I.

Ex. 23. $U = (x^2 + y^2 - 2)\,\Omega^3 + 3\,(x^2 - y^2 - 1)\,\Omega^2 + 6\,(y^2 - 1)\,\Omega + x^2 + y^2 - 4 = 0.$

Fig. 21,

$$H = (-x^4 + 4x^2y^2 + y^4 - 8y^2 + 3)\,\Omega^2 + (x^4 + 3y^4 - 4x^2 - 6y^2 + 6)\,\Omega + x^4 - 5y^4 + 11y^2 - 5x^2 = 0,$$

$$\Delta = 4\,(-x^4 + 4x^2y^2 + y^4 - 8y^2 + 3)\,(x^4 - 5y^4 + 11y^2 - 5x^2) - x^4 + 3y^4 - 4x^2 - 6y^2 + 6 = 0.$$

$$U_1 = 5\,(x^2 + y^2 - 3) = 0.$$

This curve, U_1, is the conic passing through the four cusps N, and touching the cuspidal tangents.

$$H_1 = \{2x^2 + 2\,(2 + \sqrt{5})\,y^2 - 9 - 3\sqrt{5}\}\,\{2x^2 + 2\,(2 - \sqrt{5})\,y^2 - 9 + 3\sqrt{5}\} = 0.$$

Thus the corresponding curve of FH splits up into two conics, one through the cusps M and N, the other through the cusps L and N.

Ex. 24. $U = (x^2 - 2y^2 + 1)\,\Omega^3 + 3\,(x^2 - y^2)\,\Omega^2 + 3\,(x^2 + y^2 - 1)\,\Omega - 8y^2 + 2 = 0.$ Fig. 22,

$$H = (x^2y^2 - 3y^4 + 3y^2 - 1)\,\Omega^2 + (-x^4 - 8x^2y^2 + 17y^4 + 3x^2 - 13y^2 + 2)\,\Omega$$
$$- (x^4 + 10x^2y^2 - 7y^4 - 4x^2 + 1) = 0,$$

$$\Delta = 4\,(x^2y^2 - 3y^4 + 3y^2 - 1)\,(x^4 + 10x^2y^2 - 7y^4 - 4x^2 + 1)$$
$$+ (x^4 + 8x^2y^2 - 17y^4 - 3x^2 + 13y^2 - 2)^2 = 0.$$

In this example the cusps M, N are imaginary; and there is a dp on Δ at the origin. The U and H at this point are

$$U_1 = 7x^2 - 10y^2 = 0,$$
$$H_1 = 2x^4 + 17x^2y^2 - 21y^4 - 7x^2 + 10y^2 = 0.$$

A complete investigation of the different exceptional cases that can occur in the cubic family of conics, *e.g.*, of the occurrence of fixed points, P.I.'s, line-pairs, &c., is not attempted here. Since a family of conics cannot have a locus of triple-points, nodes or cusps, any common factor of a, b and c must be either an osc-envelope or a P.I.

Families of cubics, quartics, &c., can be investigated in the same way, and some examples have already been given.

IV. The Quartic Family.

Notation.

§ 37. The principal invariants and covariants of the quartic family may be found in Professor Cayley's *Fifth Memoir upon Quantics* (Collected Papers, Vol. II., No. 156, § 128).

(1) The general integral equation is

$$U \equiv a\Omega^4 + 4b\Omega^3 + 6c\Omega^2 + 4d\Omega + e = 0,$$

where a, b, c, d, e are algebraic functions of x, y.

(2) To simplify the analysis and to bring out the connection with the cubic family, it is convenient to write

$$\mathrm{a} = ac - b^2,$$
$$\mathrm{b} = ad - bc,$$
$$\mathrm{c} = bd - c^2,$$
$$\mathrm{d} = be - cd,$$
$$\mathrm{e} = ce - d^2,$$

and $$\mathrm{g} = ae - c^2.$$

We then have the quadrinvariant

$$I = ae - 4bd + 3c^2 = \mathrm{g} - 4\mathrm{c}.$$

(3) The cubinvariant

$$J = ace + 2bcd - ad^2 - be^2 - c^3 = \begin{vmatrix} a, & & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

Since e is the minor of a in this determinant, d the minor of b, and so on,

$$J^2 = \begin{vmatrix} \mathrm{e}, & \mathrm{d}, & \mathrm{c} \\ \mathrm{d}, & \mathrm{g}, & \mathrm{b} \\ \mathrm{c}, & \mathrm{b}, & \mathrm{a} \end{vmatrix}.$$

We have therefore

$$\begin{aligned} J &= \quad a\mathrm{e} - b\mathrm{d} + c\mathrm{c} \\ &= - b\mathrm{d} + c\mathrm{g} - d\mathrm{b} \\ &= - c\mathrm{c} - d\mathrm{b} + e\mathrm{a}, \end{aligned}$$

and
$$a\mathrm{c} - b\mathrm{b} + c\mathrm{a} = 0, \text{ \&c.};$$

also
$$\begin{aligned} cJ &= \mathrm{ae} - \mathrm{c}^2, \\ aJ &= \mathrm{ag} - \mathrm{b}^2, \\ eJ &= \mathrm{eg} - \mathrm{d}^2. \end{aligned}$$

(4) The Hessian is the quartic family,

$$H = \mathrm{a}\Omega^4 + 2\mathrm{b}\Omega^3 + (\mathrm{g} + 2\mathrm{c})\,\Omega^2 + 2\mathrm{d}\Omega + \mathrm{e} = 0.$$

(5) The cubicovariant is the sextic family,

$$\Phi \equiv (a', b', c', d', e', f', g')\,(\Omega, 1)^6 = 0,$$

where
$$\begin{aligned} a' &= - a\mathrm{b} + 2b\mathrm{a}, \\ b' &= - a\mathrm{g} - 2a\mathrm{c} + 6c\mathrm{a}, \\ c' &= - 5a\mathrm{d} + 10d\mathrm{a}, \\ d' &= \quad 10\,(- a\mathrm{e} + e\mathrm{a}), \\ e' &= \quad 5e\mathrm{b} - 10b\mathrm{e}, \\ f' &= \quad e\mathrm{g} + 2e\mathrm{c} - 6c\mathrm{e}, \\ g' &= \quad e\mathrm{d} - 2d\mathrm{e}. \end{aligned}$$

(6) The Ω-discriminant is

$$\Delta \equiv I^3 - 27J^2.$$

It should be noticed that the discriminant is the same as that of Euler's cubic, except as to a power of a.

(7) The first derived functions are

$$X = a\Omega^3 + 3b\Omega^2 + 3c\Omega + d = 0,$$

and

$$Y = b\Omega^3 + 3c\Omega^2 + 3d\Omega + e = 0.$$

Let

$$X' = a\Omega^2 + 2b\Omega + c,$$

$$Y' = b\Omega^2 + 2c\Omega + d,$$

$$Z' = c\Omega^2 + 2d\Omega + e.$$

Hence

$$U = X\Omega + Y = 0,$$

$$U = X'\Omega^2 + 2Y'\Omega + Z' = 0.$$

It is convenient to call the envelope of X, which is a cubic family, Δ_1; and that of Y, Δ_2.

(8) The general p-equation,

$$V' \equiv Ap^4 + 4Bp^3 + 6Cp^2 + 4Dp + E = 0,$$

is the eliminant of Ω between the equations $U = 0$ and $\dfrac{dU}{dx} = 0$.

Using (ab_x) to denote $ab_x - ba_x$, the required eliminant is (see Salmon's *Higher Algebra*, § 84)

$$\begin{vmatrix} (ab_x) + p\,(ab_y), & (ac_x) + p\,(ac_y) & , \ldots, \ldots \\ (ac_x) + p\,(ac_y), & (ad_x) + (bc_x) + p\,(ad_y) + p\,(bc_y), & \ldots, \ldots \\ (ad_x) + p\,(ad_y), & (ae_x) + (bd_x) + p\,(ae_y) + p\,(bd_y), & \ldots, \ldots \\ (ae_x) + p\,(ae_y), & (be_x) + p\,(be_y) & , \ldots, \ldots \end{vmatrix}.$$

Hence

$$A = \begin{vmatrix} (ab_y), & (ac_y), & (ad_y), & (ae_y) \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \end{vmatrix};$$

$$B = \&c.$$

General Properties of a, b, c, &c.

§ 38. Owing to the resemblance in form between the Hessian and cubicovariant of the quartic and those of the cubic, many of the results proved for the cubic family can be extended at once to the quartic family.

It will be noticed that a, b, c are the same functions of a, b, c, d as in the cubic family, therefore the results proved above hold, and a, b, c have $3m^2$ common points. By symmetry the same thing follows for c, d, e.

Again, b, g, d have $3m^2$ points in common; for, looking at the expressions for J,

$$aJ = \text{ag} - \text{b}^2,$$

$$eJ = \text{eg} - \text{d}^2,$$

we see that J touches g at all its intersections with b, except (ac), *i.e.*, at $3m^2$ points; and at all its intersections with d except (ec), *i.e.*, at $3m^2$ points. But J is a $3m$-ic and g a $2m$-ic, therefore J cannot touch g in more than $3m^2$ points altogether; hence the intersections of b with g must be the same as the intersections of d with g, *i.e.*, b, d and g have $3m^2$ common points.

The Discriminant.

§ 39. From the equation $\Delta \equiv I^3 - 27J^2$ we see that the discriminant has cusps at all the intersections of I and J, that is, at $6m^2$ points. It lies in the region where I is positive, therefore all the cusps lie on the positive side of I and touch J. The discriminant is a $6m$-ic.

Corresponding curves of FU and FH touch the common part of their envelope on opposite sides at the same point.

When $U = 0$ has a pair of equal roots, $H = 0$ has the same pair of equal roots, hence it can be seen at once, as in the cubic family, (§ 25, p. 349) that corresponding curves of FU and FH touch their envelope at the same point. We must now determine whether they touch on the same side or not.

We know (Salmon, *Higher Algebra*, p. 297) that at every point in the region where Δ is negative $U = 0$ has two real and two imaginary roots in Ω; and since $\overline{\Delta}H = \frac{J^2\Delta}{16}$, the discriminant of H has the same sign as Δ, therefore $H = 0$ has two real and two imaginary roots in Ω at every point of the region Δ negative. At every point of the region Δ positive $U = 0$ may have four real roots *or* four imaginary roots, and similarly for $H = 0$. Now consider a point on $\Delta = 0$. $U = 0$ has two equal roots and, by a linear transformation, we can make these $\Omega = \infty$. Take the point as origin and the tangent to the curve at this point as axis of x.

Then as in the cubic family (§ 27) the equations become

$$U = (a_2 y + a_{20} x^2 + a_{11} xy + \ldots)\,\Omega^4 + 4\,(b_1 x + b_2 y + b_{20} x^2 + \ldots)\,\Omega^3$$
$$+ 6\,(c_0 + c_1 x + c_2 y + \ldots)\,\Omega^2 + 4\,(d_0 + d_1 x + d_2 y + \ldots)\,\Omega$$
$$+ e_0 + e_1 x + e_2 y + \ldots = 0,$$

$$H = \{a_2 c_0 y + (a_{20} c_0 - b_1^2)\,x^2 + \ldots\}\,\Omega^4 + 2\,(-b_1 c_0 x + \ldots)\,\Omega^3$$
$$+ (-3c_0^2 + 2b_1 d_0 x + a_2 e_0 y + \ldots)\,\Omega^2 + 2\,(-c_0 d_0 + b_1 e_0 x + \ldots)\,\Omega$$
$$+ c_0 e_0 - d_0^2 + (\quad)\,x + \ldots = 0.$$

The remaining two roots of $U = 0$ at the origin are given by

$$6c_0\,{}^2 + 4d_0\Omega + e_0 = 0,$$

and the remaining two roots of $H = 0$ by

$$-3c_0^2\Omega^2 - 2c_0 d_0\Omega + c_0 e_0 - d_0^2 = 0.$$

The roots of $U = 0$ are real or imaginary according as $2d_0^2 - 3c_0 e_0$ is positive or negative, and the roots of $H = 0$ are real or imaginary according as $2d_0^2 - 3c_0 e_0$ is negative or positive.

Hence when the remaining roots of $U = 0$ are real, those of $H = 0$ are imaginary; and when the remaining roots of $U = 0$ are imaginary, those of $H = 0$ are real.

Now a pair of real roots changes from real to imaginary in passing through a point where the values are consecutive (Note § 24), therefore we have the scheme

Δ negative	$\Delta = 0$	Δ positive
U 2 real roots	U 4 real roots	U 4 real roots
H 2 ,, ,,	H 2 ,, ,,	H no ,, ,,
U 2 real roots	U 2 real roots	U no real roots
H 2 ,, ,,	H 4 ,, ,,	H 4 ,, ,,

In the first case it is clear that U touches Δ on the positive side, H touches it on the negative side, and there are no curves of FH in the positive region; in the second case U touches Δ on the negative side, H touches it on the positive side, and there are no curves of FU in the positive region.

The shape of $U_\infty = 0$ $(a = 0)$ at the origin is

$$y + \frac{a_{20}}{a_2}\,x^2 = 0;$$

the shape of $H_\infty = 0$ ($\alpha = 0$) at the origin is

$$y + \left(\frac{a_{20}}{a_2} - \frac{b_1^{\,2}}{a_2 c_0}\right) x^2 = 0;$$

and the expression for a', *i.e.*, for Φ_∞, is exactly the same as for the cubic family, hence, as before, the shape of $\Phi_\infty = 0$ at the origin is

$$y + \left(\frac{a_{20}}{a_2} - \frac{2b_1^{\,2}}{3a_2 c_0}\right) x^2 = 0;$$

therefore Φ touches H and U, and lies between them. It can, moreover, be easily shown that Δ lies between H and Φ, and that the order in which the curves are arranged is U, Φ, Δ, H.

The similarity of the equations enables us to state for the quartic family the following results already proved for the cubic family.

Every curve of $F\Phi$ has a node on the envelope with one branch touching the envelope and the corresponding curves of FU and FH; the curves Φ and U lie on one side, and H on the opposite side of the envelope (§ 27).

The Hessian has a crunode where the corresponding curve of the cubic family has an acnode, and conversely (§ 25).

The Hessian and the corresponding curve of the cubic family have cusps at the same point, having the same tangents but turned opposite ways (§ 25).

Two curves of FH touch $J = 0$ at every point.

J^2 is a factor of the discriminant of H, and since FH has not necessarily a node-locus (*cp.* the quartic family of lines, in which FH is a non-degenerate family of conics and cannot have a node-locus) and J cannot be a P.I. for the family $H = 0$, J must be an envelope *twice*.

In the region Δ positive we may have (i) four imaginary curves of FH through every point on each side of J, and therefore J has contacts with imaginary curves; (ii) four real curves of FH through every point, on one side of J and no real curves on the other side; we cannot have four real curves of FH on each side of J, for that would necessitate six curves through a point on J, which is impossible.

The region in which Δ is positive and all curves of FU and FH are imaginary must be bounded by J, for it has already been shown that we cannot get to such a region by crossing Δ. The arrangements are as shown in figs. 23, 24,

which are diagrammatic, not giving the actual appearance, but indicating the sides on which the different curves touch.

At the intersections of I and J the Hessian curve has, in general, a double point.

If we have simultaneously $I=0$ and $J=0$, then *three* of the roots of the quartic, $U=0$, are equal (see Cayley, *l.c.*, §§ 143, 146). In this case

$$48H=-a^2(\omega_1-\omega_2)^2\{2(\Omega-\omega_2)^2+(\Omega-\omega_1)^2\}(\Omega-\omega_1)^2,$$

that is, two of the roots of $H=0$ are equal, and the other two imaginary. Now an intersection (IJ) is a cusp on $\Delta=0$ with its concavity turned towards the negative region of Δ; hence in passing through a point (IJ) we pass from a region where two roots are real through a point where two roots are equal to a region where again two roots are real, therefore the two equal roots must be coincident, not consecutive, that is, only one real curve of FH passes through the intersection (IJ) and this has a dp there. For the case of the family of lines this proves that the Hessian curve at (IJ), being a conic, is degenerate.

If, however, at an intersection of I and J four roots of $U=0$ are equal, then $H\equiv 0$, that is, $\mathrm{a}=0$, $\mathrm{b}=0$, $\mathrm{g}+2\mathrm{c}=0$, &c. *i.e.*, all the curves of FH pass through the point (see Ex. 25, where all the curves of FH pass through the points L', M').

The Quartic Family of Lines.

§ 39. In this family a, b, c, &c., are conics.

The quadratic family $\mathrm{a}\Omega^2+\mathrm{b}\Omega+c=0$, is the Hessian of the family $X, =a\Omega^3+3b\Omega^2+3c\Omega+d=0$, and the three common points of a, b, c are the cusps, (L, M, N) of $\Delta_1, =4\mathrm{ac}-\mathrm{b}^2, =0$. Similarly, $\mathrm{c}\Omega^2+\mathrm{d}\Omega+\mathrm{e}=0$ is the Hessian of

$$Y, =b\Omega^3+3c\Omega^2+3d\Omega+e, =0$$

and c, d, e have three common points which we may call (L', M', N').

$I=0$ is the conic $\mathrm{g}-4\mathrm{c}=0$.

$J=0$ is a cubic touching a at the three points (L, M, N) and e at the three points (L', M', N'). This follows from the equations

$$cJ=\mathrm{ae}-\mathrm{c}^2, \text{ \&c.}$$

Also J lies in the regions where a and c have the same sign.

The Discriminant.

$\Delta, \equiv I^3-27J^2, =0$, being the envelope of a family of

lines, is a unicursal sextic of class 4. From its equation we see that it has six cusps touching $J=0$ at the six intersections (IJ), hence it must have four dps, three dts and no inflexions. It lies entirely on the positive side of $I=0$.

A linear construction for the discriminant is easily found from the theory of the cubic family.

We know that X', $=a\Omega^2+2b\Omega+c$, $=0$, is a tangent to a and $Y'=0$ is the corresponding tangent to c. The intersection of X' and Y' lies on Δ_1, the envelope of X, and since X, $=X'\Omega+Y'$, $=0$ is the tangent to Δ_1 at the point $(X'Y')$ it can be drawn at once; similarly the corresponding Y can be drawn. But the corresponding X and Y intersect on the required sextic, hence the sextic can be drawn.

Ex. 25. $U=(2x-1)\Omega^4+4(x+y)\Omega^3+6y\Omega^2+4(x-y)\Omega-2x-1=0.$ Fig. 25

$$I=-8x^2+7y^2+1,$$

$$J=(2x^2-y^2+y),$$

$$\Delta=(-8x^2+7y^2+1)^3-27(2x^2-y^2+y)^2(3y+1)^2.$$

$\Delta=0$ is a unicursal sextic of class 4. It has two cusps L, M at the points $(-1,-1)$, $(1,-1)$; singularities equivalent to two cusps and a dp at each of the points L', M', where I passes through a dp on J; and dps at N, N', making six cusps and four dps.

The Hessian at the cusp M splits up into the two lines ML', MM'. All the Hessian curves pass through the points L', M'.

Ex. 26. $U=(x+y+2)\Omega^4+4(x-y+1)\Omega^3+6y\Omega^2-4(x+y-1)\Omega$
$-x+y+2=0.$ Fig. 26

$$I=3(x^2+4y),$$

$$J=9x^2y+12y-4,$$

$$\Delta=27\{(x^2+4y)^3-(-9x^2y+12y-4)^2\}.$$

The form of the sextic, $\Delta=0$ is shown in the diagram. At the point P the curves a and a touch Δ on opposite sides. There is a region bounded by $J=0$ in which there are no real curves of FU or FH.

Summary.

§ 40. We close the discussion with the quartic family of lines. To proceed further by the methods used above, and to give examples would involve very troublesome analysis. It is, however, easy to see how the investigation would proceed.

We have shown that the family of curves represented by the *general* rational integral algebraic equation of the n^{th} degree in one arbitrary parameter Ω, which has as coefficients algebraic functions of x, y, has in every case an envelope.

This envelope is the common factor of the Ω-discriminant of the integral equation and the p-discriminant of the corresponding differential equation. The remaining factor of the p-discriminant is the square of the tac-locus.

If, however, the Ω-equation is *specialised*, two or more branches of the envelope may coincide and become

(i) the locus of a singularity,

(ii) a P.I., *i.e.* branches of two or more distinct, coincident, or consecutive curves of the family.

In either case one or more branches of the tac-locus will coincide with the coincident branches of the envelope.

By considering the number of curves and of branches of curves through every point of the plane, we have found a limit to the order and nature of the singularities whose loci can occur in any particular family, also to the combinations of loci which can occur. The number of fixed points which can occur in the family has been found, and the order of the singularity on the envelope which corresponds to a fixed point of the family is given.

It is next proved that the number of contacts of each curve of the family with its envelope depends, in general, only on the degree of the curves of the family, not on the value of n.

The quadratic family of curves is then discussed in detail, and, from our knowledge of the singularity-loci and forms of P.I.'s which can occur as factors of the Ω-discriminant, &c., it has been found possible to extend Professor Casorati's table so as to show at a glance the geometrical function of any factor of the Ω and p-discriminants.

In the cubic family we first meet with some covariants of the Ω-equation, namely, the Hessian and cubicovariant. It is easily proved that the Hessian family and the original cubic family have the same Ω-discriminant, and that the factors of this have the same functions in each family. Now the Hessian is a quadratic family, consequently cannot have singularities and combinations of loci of so high an order as the cubic family. It has been shown that these higher singularity-loci and combinations of loci appear as factors which divide out when the Hessian family is formed. By means of this the geometrical functions of the factors of the discriminants of the cubic family can be determined.

Since the Hessian of the quartic family is itself a quartic family, the discussion of it does not prove so useful in simplifying the determination of the geometrical properties of the

Ω-discriminant. It has however been shown that the geometrical relations of the Hessian and cubicovariant to the original family are the same both for the cubic and quartic families, and it seems clear that, in the case of the Hessian, this must hold for the family of the n^{th} degree. The proof of this, and possibly an investigation of other invariants and covariants, are left for another time.

Bryn Mawr College,
Pennsylvania,
September, 1894.

Note.

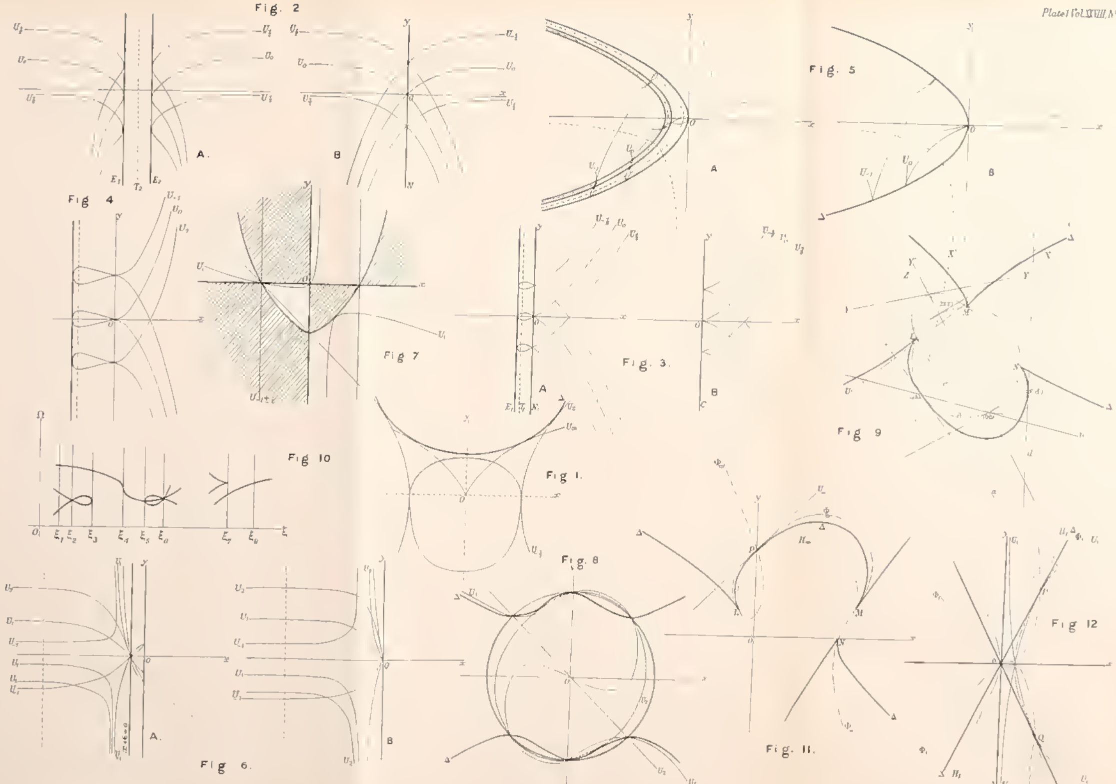
Since the above was written an article on the earlier part of the same subject has appeared in the *Giornale di matematiche di Battaglini*, Vol. XXXIII. (ii. of 2nd Series), pp. 31—56, 183—209, entitled "Sulle solutioni singolari delle equazioni differenziali ordinarie di 1° ordine."

The author, Lia Predella, gives an account of the literature of the subject, discusses Mr. Workman's treatment of the superposition of loci, investigates the equation of the first degree in p, and deals with the quadratic family and a simplified form of the cubic family by Casorati's methods. The discussion is hardly satisfactory; for instance, there is an error in the treatment of P.I.'s in the quadratic family on p. 195, where the writer, taking h as a factor of Δ, says, "suppose that h does not divide any of the three functions a, b, c; then $h = 0$ is not a solution."

That $h = 0$ may be a P.I. in this case is easily seen from §§ 14 and 15; for there is no reason why a P.I. which occurs in two distinct or consecutive curves should belong to the special curves U_0 or U_∞ (see Ex. 10, where $y = 0$ is a P.I. which does not divide a, b, or c).

Another addition to the bibliography of the subject, is a paper by Professor M. J. M. Hill, "On the flex-locus of a system of plane curves whose equation is a rational integral function of the coordinates and one arbitrary parameter", *Messenger of Mathematics*, Vol. XXIII., pp. 120-9, 1893-94.

December, 1895.



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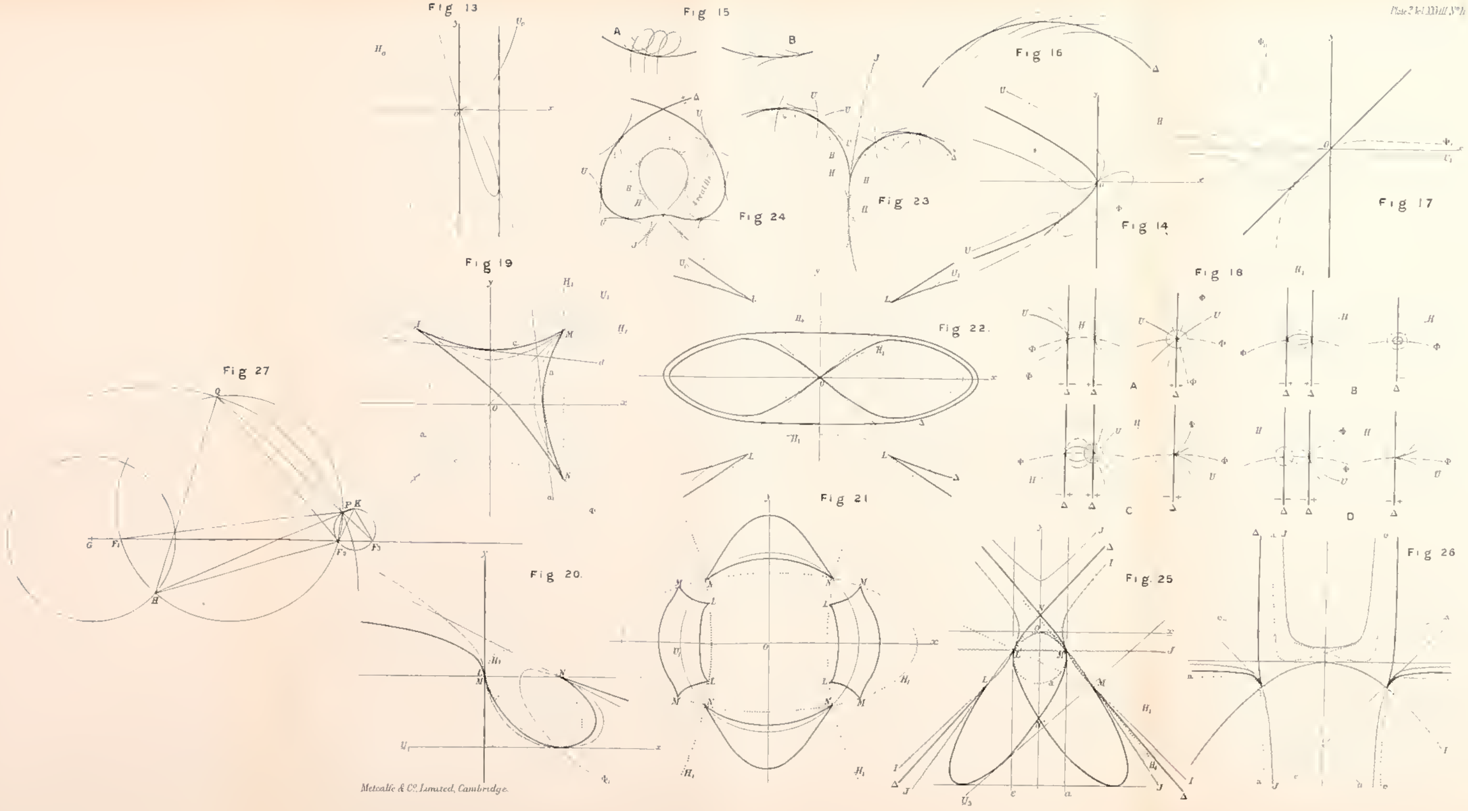

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